

# Transverse Centroid and Envelope Descriptions of Beam Evolution

\* Research supported by the US Dept. of Energy at LLNL and LBNL under contract Nos. DE-AC52-07NA27344 and DE-AC02-05CH11231.

Steven M. Lund

Lawrence Livermore National Laboratory (LLNL)

Steven M. Lund and John J. Barnard

“Beam Physics with Intense Space-Charge”

US Particle Accelerator School

University of Maryland, held at Annapolis, MD

16-27 June, 2008

(Version 20080624)

\* Research supported by the US Dept. of Energy at LLNL and LBNL under contract Nos. DE-AC52-07NA27344 and DE-AC02-05CH11231.

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 1

## Transverse Centroid and Envelope Model: Outline

### Overview

Derivation of Centroid and Envelope Equations of Motion

Centroid Equations of Motion

Envelope Equations of Motion

Matched Envelope Solutions

Envelope Perturbations

Envelope Modes in Continuous Focusing  
Envelope Modes in Periodic Focusing

Transport Limit Scaling Based on Envelope Models

Centroid and Envelope Descriptions via 1<sup>st</sup> order Coupled Moment Equations  
Comments:  
◆ Some of this material related to J.J. Barnard lectures:

- Transport limit discussions ([Introduction](#))
- Transverse envelope modes ([Continuous Focusing Envelope Modes and Halo](#))
  - Longitudinal envelope evolution ([Longitudinal Beam Physics III](#))
  - 3D Envelope Modes in a Bunched Beam ([Cont. Focusing Envelope Modes and Halo](#))
- ◆ Specific topics will be covered in more detail here

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 2

## Transverse Centroid and Envelope Model: Detailed Outline

### 1) Overview

### 2) Derivation of Centroid and Envelope Equations of Motion

Statistical Averages

Particle Equations of Motion

Distribution Assumptions

Self-Field Calculation: Direct and Image

Coupled Centroid and Envelope Equations of Motion

### 3) Centroid Equations of Motion

Single Particle Limit: Oscillation and Stability Properties

Effect of Driving Errors

Effect of Image Charges

### 4) Envelope Equations of Motion

KV Envelope Equations

Applicability of Model

Properties of Terms

### 5) Matched Envelope Solution

Construction of Matched Solution

Symmetries of Matched Envelope: Interpretation via KV Envelope Equations  
Examples

## Detailed Outline - 2

### 6) Envelope Perturbations

Perturbed Equations

Matrix Form: Stability and Mode Symmetries

Decoupled Modes

General Mode Limits

### 7) Envelope Modes in Continuous Focusing

Normal Modes: Breathing and Quadrupole Modes

Driven Modes

### 8) Envelope Modes in Periodic Focusing

Solenoidal Focusing

Quadrupole Focusing

Launching Conditions

### 9) Transport Limit Scaling Based on Envelope Models

Overview

Example for a Periodic Quadrupole FODO Lattice

Discussion and Application of Formulas in Design

Results of More Detailed Models

## Detailed Outline - 3

### 10) Centroid and Envelope Descriptions via 1<sup>st</sup> Order Coupled Moment

#### Equations

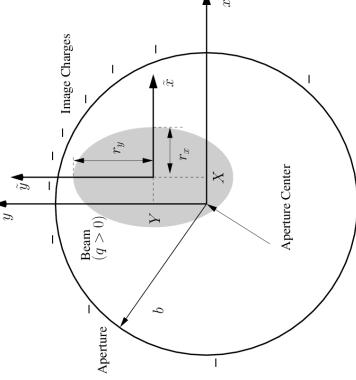
Formulation

Example Illustration -- Familiar KV Envelope Model

#### Contact Information

#### References

Analyze transverse centroid and envelope properties of an unbunched ( $\partial/\partial z = 0$ ) beam



**Centroid:**

$$X = \langle x \rangle_{\perp}$$

$$Y = \langle y \rangle_{\perp}$$

**Envelope:** (edge measure)

$$r_x = 2\sqrt{\langle (x - X)^2 \rangle_{\perp}}$$

$$r_y = 2\sqrt{\langle (y - Y)^2 \rangle_{\perp}}$$

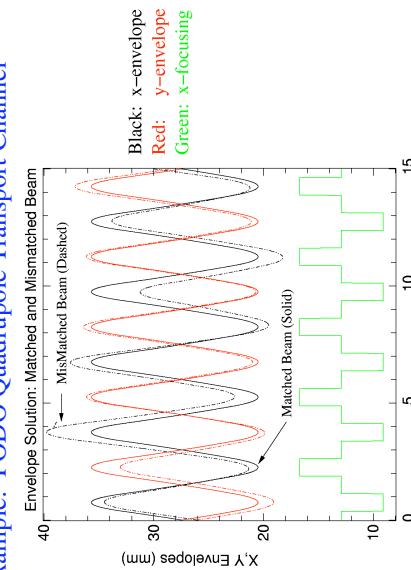
SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 5

$$\langle \dots \rangle_{\perp} \equiv \frac{\int d^2x_{\perp} \int d^2x'_{\perp} \dots f_{\perp}}{\int d^2x_{\perp} \int d^2x'_{\perp} f_{\perp}}$$

Transverse averages:

Mismatched beams have larger envelope excursions and have more stability problems since mismatch adds another source of free energy that can drive statistical increases in particle amplitudes (see: J.J. Barnard lectures on [Envelopes and Halo](#))

**Example:** FODO Quadrupole Transport Channel



Oscillations in the statistical beam centroid and envelope radii are the *lowest-order* collective responses of the beam

**Centroid Oscillations:** Associated with errors and are purposefully suppressed to the level possible

◆ Error Sources:

- Beam distribution assymmetries
- Dipole bending terms from applied field optics
- Imperfect mechanical alignment
- ◆ Exception: When the beam is kicked (insertion or extraction) into or out of a transport channel as is often done in rings

**Envelope Oscillations:** Can have two components in periodic focusing lattices

- 1) **Matched Envelope:** Periodic "flutter" synchronized to period of focusing lattice to yield net focusing
  - ◆ Properly tuned flutter essential in Alternating Gradient quadrupole lattices
- 2) **Mismatched Envelope:** Excursions deviate from matched flutter motion and are seeded/driven by errors

Limiting maximum beam-edge excursions is desired for economical transport

- Reduces cost by Limiting material volume needed to transport an intense beam

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 7

◆ Larger machine aperture is needed to confine a mismatched beam

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 8

## Centroid and Envelope oscillations are the *most important collective modes* of an intense beam

- Force balances based on matched beam envelope equation predict scaling of transportable beam parameters
  - Used to design transport lattices
- Instabilities in beam centroid and/or envelope oscillations prevent reliable transport
  - Parameter locations of instability regions should be understood and avoided in machine design/operation

Although it is *necessary* to design to avoid envelope and centroid instabilities, it is not alone *sufficient* for effective machine operation

- Higher-order kinetic and fluid instabilities not expressed in the low-order envelope models can degrade beam quality and control and must also be evaluated
- To be covered (see: S.M. Lund, lectures on [Kinetic Stability](#))

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 9

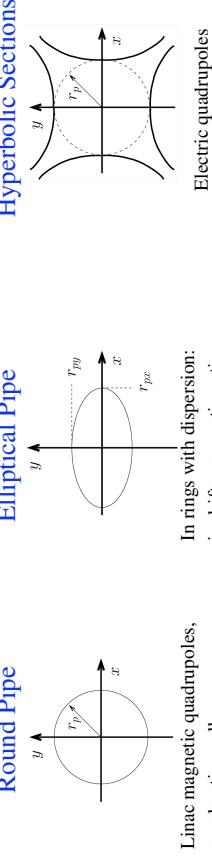
## Transverse Particle Equations of Motion

Consistent with earlier analysis, take:

$$\begin{aligned} x'' + \frac{(\gamma_b \beta_b)' x' + \kappa_x x}{m \gamma_b^3 \beta_b^2 c^2} &= -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x} && \text{Assume:} \\ y'' + \frac{(\gamma_b \beta_b)' y' + \kappa_y y}{m \gamma_b^3 \beta_b^2 c^2} &= -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial y} && \begin{array}{l} \text{Unbunched beam} \\ \text{No axial momentum spread} \\ \text{Linear applied focusing fields} \end{array} \\ \nabla_{\perp}^2 \phi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \phi &= -\frac{\rho}{\epsilon_0} && \begin{array}{l} \text{described by } \kappa_x, \kappa_y \\ \text{Possible acceleration } \gamma_b \beta_b \\ \text{need not be constant} \end{array} \\ \rho = q \int d^2 x_{\perp} f_{\perp} &\quad \phi|_{\text{aperture}} = 0 && \end{array}$$

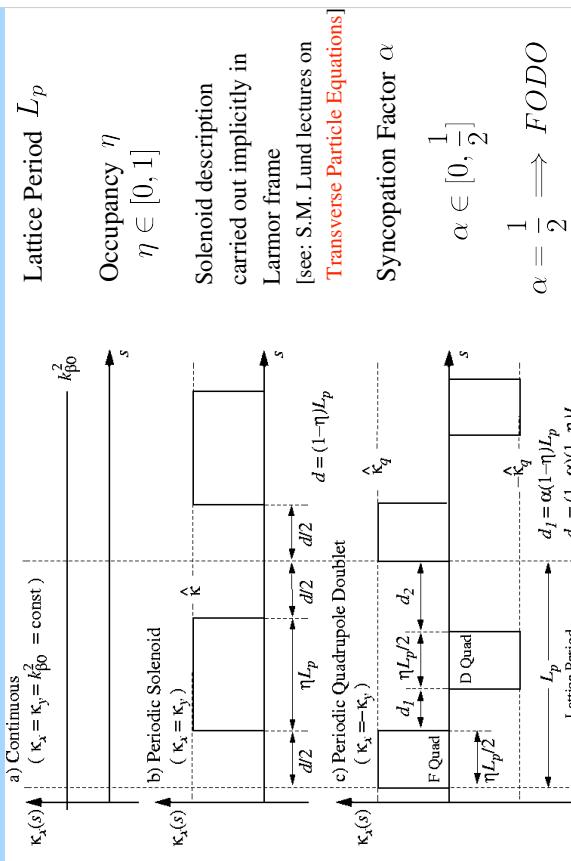
Various apertures are possible. Some simple examples:

### Round Pipe



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 11

## Review: Focusing lattices we will take in examples: Continuous and piecewise constant periodic solenoid and quadrupole doublet



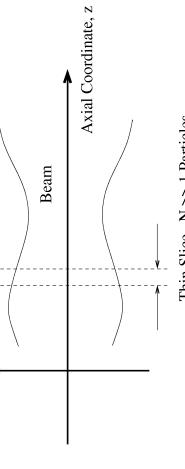
SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 12

## S2: Derivation of Transverse Centroid and Envelope Equations of Motion

Analyze centroid and envelope properties of an unbunched ( $\partial/\partial z = 0$ ) beam

### Transverse Statistical Averages:

- Let  $N$  be the number of particles in a thin axial slice of the beam at axial coordinate  $s$ .
- Thin Slice,  $N \gg 1$  Particles



Equivalent averages can be defined in terms of the particles or the transverse Vlasov distribution function:

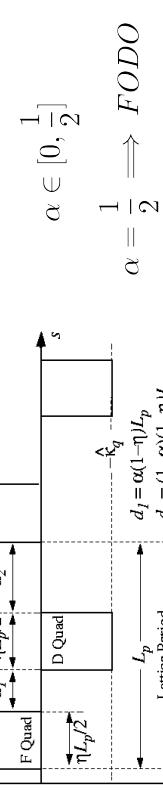
$$\begin{aligned} \text{particles: } \langle \dots \rangle_{\perp} &\equiv \frac{1}{N} \sum_{i=1}^N \dots \\ \text{distribution: } \langle \dots \rangle_{\perp} &\equiv \frac{\int d^2 x_{\perp} f_{\perp}'}{\int d^2 x_{\perp} f_{\perp}} \dots \end{aligned}$$

- Averages can be generalized to include axial momentum spread

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 10

$$\begin{aligned} \text{Occupancy } \eta &\\ \eta \in [0, 1] & \\ \text{Solenoid description carried out implicitly in Larmor frame} & \\ [\text{see: S.M. Lund lectures on Transverse Particle Equations}] & \end{aligned}$$

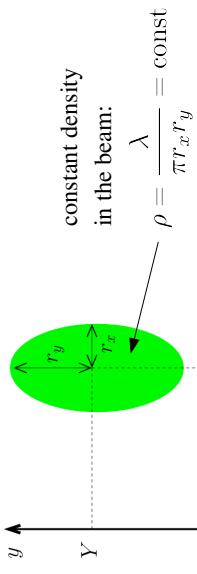
$$\begin{aligned} \text{Syncopation Factor } \alpha & \\ \alpha \in [0, \frac{1}{2}] & \\ \alpha = \frac{1}{2} \implies FODO & \end{aligned}$$



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 12

## Distribution Assumptions

To lowest order, linearly focused intense beams are expected to be nearly uniform in density within the core of the beam out to an edge where the density falls rapidly to zero



- ♦ Expected for near equilibrium structure for strong space-charge due to Debye screening (see: S.M. Lund, lectures on [Transverse Equilibrium Distributions](#))
- ♦ Observed in simulations of stable non-equilibrium beams

$$\rho(x, y) = q \int d^2 x'_\perp f_\perp \simeq \begin{cases} -\frac{\lambda}{\pi r_x r_y}, & (x - X)^2/r_x^2 + (y - Y)^2/r_y^2 < 1 \\ 0, & (x - X)^2/r_x^2 + (y - Y)^2/r_y^2 > 1 \end{cases}$$

$$\lambda = q \int d^2 x'_\perp \int d^2 x'_\perp f_\perp = \int d^2 x \rho$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 13

## Self-Field Calculation

Temporarily, we will consider an arbitrary beam charge distribution within an arbitrary aperture to formulate the problem.

**Electrostatic field of a line charge in free-space**

$$\mathbf{E}_\perp = \frac{\lambda_0}{2\pi\epsilon_0} \frac{(\mathbf{x}_\perp - \tilde{\mathbf{x}})}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}|^2} \quad \mathbf{x}_\perp = \tilde{\mathbf{x}} = \text{coordinate of charge}$$

Resolve the field of the beam into direct (free space) and image terms:

$$\mathbf{E}_\perp^s = -\frac{\partial\phi}{\partial\mathbf{x}_\perp} = \mathbf{E}_\perp^d + \mathbf{E}_\perp^i$$

and superimpose free-space solutions for direct and image contributions

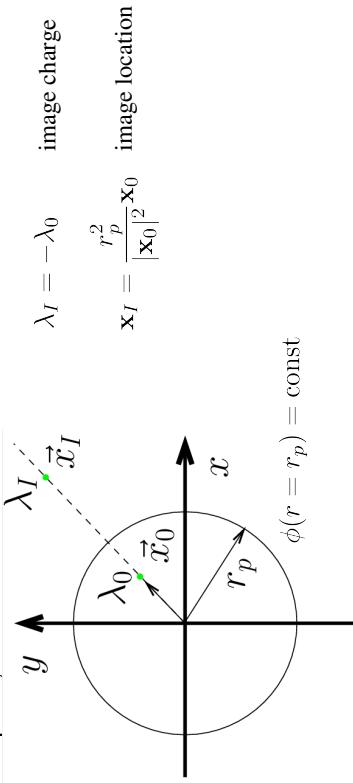
$$\mathbf{E}_\perp^d = \frac{1}{2\pi\epsilon_0} \int d^2 \tilde{\mathbf{x}}_\perp \frac{\rho(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2} \quad \rho(\mathbf{x}) = \text{beam charge density}$$

$$\mathbf{E}_\perp^i = \frac{1}{2\pi\epsilon_0} \int d^2 \tilde{\mathbf{x}}_\perp \frac{\rho^i(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2} \quad \rho^i(\mathbf{x}) = \text{beam image charge density induced on aperture}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 14

## Direct Field:

Image structure depends on the aperture. Assume a round pipe (most common case) for simplicity.



superimpose all images of beam:

$$\mathbf{E}_\perp^i(\mathbf{x}_\perp) = -\frac{1}{2\pi\epsilon_0} \int_{\text{pipe}} d^2 \tilde{\mathbf{x}}_\perp \frac{\rho(\tilde{\mathbf{x}}_\perp)(\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp)^2 / |\tilde{\mathbf{x}}_\perp|^2}{|\mathbf{x}_\perp - \tilde{\mathbf{x}}_\perp|^2 / |\tilde{\mathbf{x}}_\perp|^2}$$

- ♦ Difficult to calculate even for  $\rho$  corresponding to a uniform density beam

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 15

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 16

Examine limits of the image field to build intuition on the range of properties:

### 1) On-axis line charge:

$$\mathbf{E}_\perp^i(\hat{\mathbf{x}}_\perp = X\hat{\mathbf{x}}) = \frac{\lambda}{2\pi\epsilon_0(r_p^2/X - X)}\hat{\mathbf{x}}$$

- ◆ Generates nonlinear field at position of direct charge
- ◆ Field creates attractive force between direct and image charge

### 2) Centered, uniform density elliptical beam:

Expand using complex coordinates starting from the general image expression:

$$\begin{aligned} \underline{E}_y^i = E_y^i + iE_x^i &= \sum_{n=2,4,\dots}^{\infty} c_n z^{n-1} = \frac{i}{2\pi\epsilon_0} \int_{\text{pipe}} d^2x_\perp \rho(\mathbf{x}_\perp) \frac{(x - iy)^n}{r_p^{2n}} \\ &= \frac{i\lambda n!}{2\pi\epsilon_0 2^n (n/2+1)!(n/2)!} \left( \frac{r_x^2 - r_y^2}{r_p^4} \right)^{n/2} \end{aligned}$$

$$z = x + iy \quad i = \sqrt{-1}$$

The linear ( $n = 2$ ) components of this expansion give:

$$E_x^i = \frac{\lambda}{8\pi\epsilon_0} \frac{r_x^2 - r_y^2}{r_p^4} x, \quad E_y^i = -\frac{\lambda}{8\pi\epsilon_0} \frac{r_x^2 - r_y^2}{r_p^4} y$$

- ◆ Rapidly vanish (higher order terms more rapid) as beam becomes more round

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 17

### 3) Uniform density elliptical beam with a small displacement along the $x$ -axis:

$$Y = 0$$

$$|X|/r_p \ll 1$$

Expand using complex coordinates starting from the general image expression:

- ◆ Use complex coordinates to simplify calculation

E.P. Lee, E. Close, and L. Smith, Nuclear Instruments and Methods, 1126 (1987)

- ◆ Expressions become even more complicated with simultaneous  $x$ - and  $y$ -displacements and more complicated aperture geometries

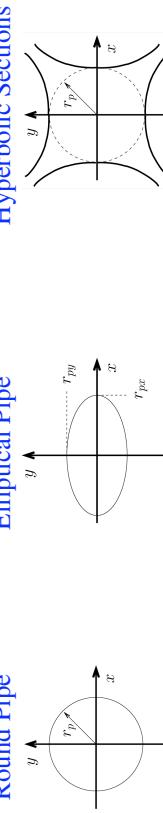
Leading Order	$E_x^i = \frac{\lambda}{2\pi\epsilon_0 r_p^2} [f(x - X) + gX] + \Theta\left(\frac{X}{r_p}\right)^3$
Image Fields	$E_y^i = -\frac{\lambda}{2\pi\epsilon_0 r_p^2} fy + \Theta\left(\frac{X}{r_p}\right)^3$
	$f = \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{X^2}{r_p^2} \left[ 1 + \frac{3}{2} \left( \frac{r_x^2 - r_y^2}{r_p^2} \right) + \frac{3}{8} \left( \frac{r_x^2 - r_y^2}{r_p^2} \right)^2 \right]$
	$g = 1 + \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{X^2}{r_p^2} \left[ 1 + \frac{3}{4} \left( \frac{r_x^2 - r_y^2}{r_p^2} \right) + \frac{1}{8} \left( \frac{r_x^2 - r_y^2}{r_p^2} \right)^2 \right]$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 18

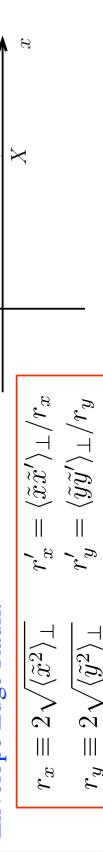
### Comments on images:

- ◆ Sign is generally such that it will tend to increase beam displacements
  - Also (usually) weak linear focusing corrections for an elliptical beam
- ◆ Can be very difficult to calculate explicitly
  - Even for simple case of circular pipe
  - Special cases of simple geometry formulas can give idea on scaling
  - Generally suppress just by making the beam small relative to characteristic aperture dimensions and keeping the beam steered near-axis
- ◆ Depend strongly on the aperture geometry
  - Generally varies as a function of  $s$  in the machine aperture changes and/or beam symmetries evolve

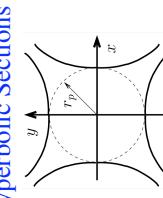
### Round Pipe



### Elliptical Pipe



### Hyperbolic Sections



### Coupled centroid and envelope equations of motion

Consistent with the assumed structure of the distribution (uniform density elliptical beam), denote:

#### Beam Centroid:

$$\begin{aligned} X &\equiv \langle x \rangle_\perp & X' &\equiv \langle x' \rangle_\perp \\ Y &\equiv \langle y \rangle_\perp & Y' &\equiv \langle y' \rangle_\perp \end{aligned}$$

#### Coordinates with respect to centroid:

$$\begin{aligned} \tilde{x} &\equiv x - X & \tilde{x}' &\equiv x' - X' \\ \tilde{y} &\equiv y - Y & \tilde{y}' &\equiv y' - Y' \end{aligned}$$

#### Envelope Edge Radii:

$$\begin{aligned} r_x &\equiv 2\sqrt{\langle \tilde{x}^2 \rangle_\perp} & r'_x &\equiv \langle \tilde{x}'^2 \rangle_\perp / r_x \\ r_y &\equiv 2\sqrt{\langle \tilde{y}^2 \rangle_\perp} & r'_y &\equiv \langle \tilde{y}'^2 \rangle_\perp / r_y \end{aligned}$$

With the *assumed* uniform elliptical beam, all moments can be calculated in terms of:  $X, Y, r_x, r_y$

- ◆ Such truncations follow whenever the form of the distribution is “frozen”

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 19

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 20

Derive centroid equations: First use the self-field resolution for a uniform density beam, then the equations of motion for a particle within the beam are:

$$\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x - \frac{2Q}{(r_x + r_y)r_x}(x - X) &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_x^i \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y - \frac{2Q}{(r_x + r_y)r_y}(y - Y) &= \frac{q}{m\gamma_b^3\beta_b^2c^2}E_y^i \end{aligned}$$

**Direct Terms**

Pervance:

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} \quad (\text{not necessarily constant if beam accelerates})$$

average equations using:  $\langle x' \rangle_\perp = \langle x \rangle'_\perp = X'$  etc., to obtain:

**Centroid Equations:**

$$\begin{aligned} X'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}X' + \kappa_x X &= Q \left[ \frac{2\pi\epsilon_0}{\lambda} \langle E_x^i \rangle_\perp \right] \\ Y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}Y' + \kappa_y Y &= Q \left[ \frac{2\pi\epsilon_0}{\lambda} \langle E_y^i \rangle_\perp \right] \end{aligned}$$

♦  $\langle E_x^i \rangle_\perp$  will generally depend on:  $X, Y$  and  $r_x, r_y$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 21

Note: the electric image field will cancel the coefficient  $2\pi\epsilon_0/\lambda$

To derive equations of motion for the envelope radii, first subtract the centroid equations from the particle equations of motion ( $\tilde{x} \equiv x - X$ ) to obtain:

$$\begin{aligned} \tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa_x \tilde{x} - \frac{2Q\tilde{x}}{(r_x + r_y)r_x} &= \frac{q}{m\gamma_b^2\beta_b^2c^2} [E_x^i - \langle E_x^i \rangle] \\ \tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa_y \tilde{y} - \frac{2Q\tilde{y}}{(r_x + r_y)r_x} &= \frac{q}{m\gamma_b^2\beta_b^2c^2} [E_y^i - \langle E_y^i \rangle] \end{aligned}$$

Differentiate the equation for the envelope radius ( $y$ -equations analogous):

$$r_x = 2\langle \tilde{x}^2 \rangle_\perp^{1/2} \longrightarrow r'_x = \frac{2\langle \tilde{x}\tilde{x}' \rangle_\perp}{\langle \tilde{x}^2 \rangle_\perp} = \frac{4\langle \tilde{x}\tilde{x}' \rangle_\perp}{r_x}$$

Define (motivated the KV equilibrium results) a statistical rms edge emittance:

$$\varepsilon_x \equiv 4\varepsilon_{x,\text{rms}} \equiv 4 [\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2]^{1/2}$$

Differentiate the equation for  $r'_x$  again and use the emittance definition:

$$\begin{aligned} r''_x &= 4 \frac{\langle \tilde{x}\tilde{x}'' \rangle_\perp}{r_x} + \frac{16[\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2]}{r_x^3} \\ &= 4 \frac{\langle \tilde{x}\tilde{x}'' \rangle_\perp}{r_x} + \frac{\varepsilon_x^2}{r_x^3} \end{aligned}$$

and then employ the equations of motion to eliminate  $\tilde{x}''$  in  $\langle \tilde{x}\tilde{x}'' \rangle_\perp$  to obtain:

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 22

**Envelope Equations:**

$$\begin{aligned} r''_x + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}r'_x + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} &= 8Q \left[ \frac{\pi\epsilon_0}{\lambda} \langle \tilde{x}E_x^i \rangle_\perp \right] \\ r''_y + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}r'_y + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} &= 8Q \left[ \frac{\pi\epsilon_0}{\lambda} \langle \tilde{y}E_y^i \rangle_\perp \right] \end{aligned}$$

♦  $\langle \tilde{x}E_x^i \rangle_\perp$  will generally depend on:  $X, Y$  and  $r_x, r_y$

**Comments on Centroid/Envelope equations (Continued):**

♦ Constant (normalized when accelerating) emittances are generally assumed

- See: S.M. Lund, lectures on **Transverse Particle Equations**

$$\beta_b, \gamma_b, \lambda \quad s\text{-variation set by acceleration schedule}$$

$$\varepsilon_{nx} = \gamma_b\beta_b\varepsilon_x = \text{const} \quad \longrightarrow \quad \text{used to calculate } \varepsilon_x, \varepsilon_y$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2}$$

**Comments on Centroid/Envelope equations:**

♦ Centroid and envelope equations are coupled and must be solved simultaneously when image terms on the RHS cannot be neglected

♦ Image terms contain nonlinear terms that can be difficult to evaluate explicitly

- Aperture geometry changes image correction

♦ The formulation is not self-consistent because a frozen form (uniform density) charge profile is assumed

- Uniform density choice motivated by KV results and Debye screening

see: S.M. Lund, lectures on **Transverse Equilibrium Distributions**

- The assumed distribution form not evolving represents a fluid model closure

### S3: Centroid Equations of Motion

#### Single Particle Limit: Oscillation and Stability Properties

Neglect image charge terms, then the centroid equation of motion becomes:

$$\begin{aligned} X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \kappa_x X &= 0 \\ Y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} Y' + \kappa_y Y &= 0 \end{aligned}$$

♦ Usual Hill's equation with generalized acceleration term

♦ Single particle form. Apply results from S.M. Lund lectures on [Transverse Particle](#)

[Equations:](#) phase amplitude methods, Courant-Snyder invariants, and stability bounds, ...

Assume that applied lattice focusing is tuned for constant phase advances and/or that acceleration is weak and can be neglected. Then single particle stability results give immediately:

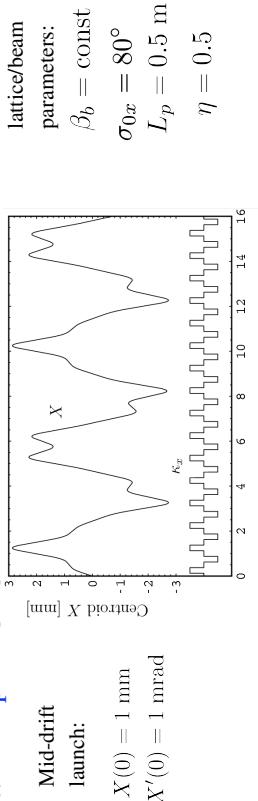
$$\begin{aligned} \sigma_{0x} < 180^\circ \\ \sigma_{0y} < 180^\circ \end{aligned}$$

centroid stability, 1<sup>st</sup> stability condition

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 25

### III

#### // Example: FODO channel centroid evolution



- ♦ Centroid exhibits expected characteristic stable betatron oscillations
- ♦ Centroid exhibits expected characteristic stable betatron oscillations

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 26

### Effect of Driving Errors

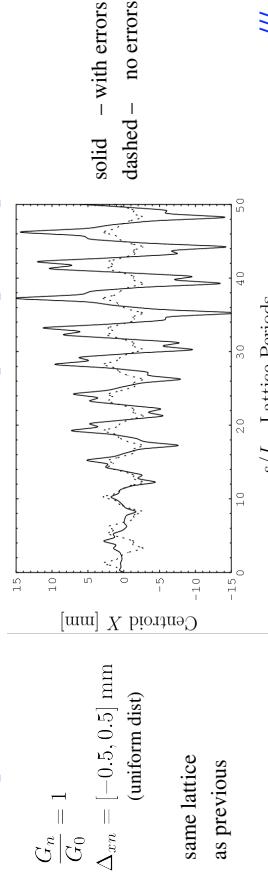
The reference orbit is **ideally tuned for zero centroid excursions**. But there will always be driving errors that can cause the centroid oscillations to accumulate with beam propagation distance:

$$X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \frac{G_n}{G_0} \kappa_q(s) X = \frac{G_n}{G_0} \kappa_q(s) \Delta_{xn}$$

$$\frac{G_n}{G_0} = \text{nth quadrupole gradient error (unity for no error; } s\text{-varying)}$$

$$\Delta_{xn} = \text{nth quadrupole transverse displacement error (} s\text{-varying)}$$

#### // Example: FODO channel centroid with quadrupole displacement errors



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 27

### III

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 28

## Effects of Image Charges

Model the beam as a displaced line-charge in a circular aperture. Then using the previously derived image charge field, the equations of motion reduce to:

$$X'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} X' + \kappa_x X = \frac{QX}{r_p^2 - X^2}$$

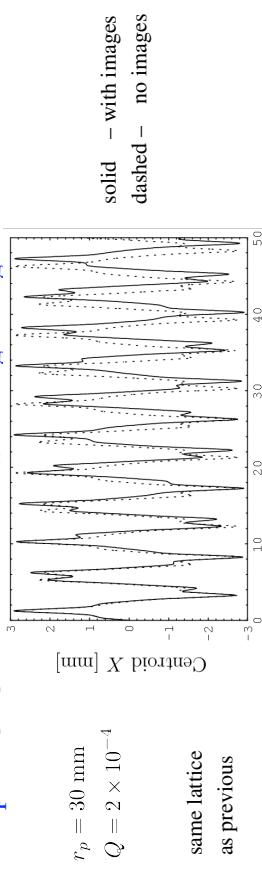
examine oscillation  
along  $x$ -axis

$$\frac{QX}{r_p^2 - X^2} \simeq \frac{Q}{r_p^2} X + \frac{Q}{r_p^4} X^3$$

linear correction

Nonlinear correction (smaller)

### Example: FODO channel centroid with image charge corrections



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 29

## S4: Envelope Equations of Motion

**Overview:** Reduce equations of motion for  $r_x$ ,  $r_y$

♦ Generally found that couplings to centroid coordinates  $X$   $Y$  are weak

- Centroid ideally zero

♦ Envelope eqns are most important in designing transverse focusing systems

- Expresses average radial force balance (see following discussion)

- Can be difficult to analyze analytically for scaling properties

- "Systems" codes generally written using envelope equations, stability criteria, and practical engineering constraints

♦ Instabilities of the envelope equations in periodic focusing lattices must be avoided in machine operation

- Instabilities are strong and real: not washed out with realistic distributions without frozen form

- Represent lowest order "KV" modes of a full kinetic theory

♦ Previous derivation of envelope equations relied on Courant-Snyder invariants in linear applied and self-fields. Analysis shows that the same force balances result for a uniform elliptical beam with no image couplings.

- Debye screening arguments suggest assumed uniform density model taken should be a good approximation for intense space-charge

Main effect of images appears to be an accumulated phase error of the centroid orbit since, generally the centroid error oscillations are not "matched" orbits. This will complicate extrapolations of errors over many lattice periods

### Control by:

- ♦ Keeping centroid displacements  $X$ ,  $Y$  small by correcting
- ♦ Make aperture (pipe radius) larger

### General Comments:

♦ Images contributions to centroid excursions generally less problematic than misalignment errors in focusing elements

More detailed analyses show that the coupling of the envelope radii  $r_x$ ,  $r_y$  to the centroid evolution in  $X$ ,  $Y$  is often weak

- ♦ Fringe fields are more important for accurate calculation of centroid orbits since orbits are not part of a matched lattice
  - Nonideal orbits are poorly tuned to lattice and become more sensitive to the precise phase of impulses
  - ♦ Over long path lengths many nonlinear terms can influence results
  - ♦ Lattice errors are not often known so one must often analyze characteristic error distributions to see if centroids measured are consistent with expectations

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 30

## KV/rms Envelope Equations: Properties of Terms

The envelope equation reflects low-order force balances:

$r_x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r'_x$	$+ \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^2} = 0$
$r_y'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r'_y$	$+ \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^2} = 0$
Applied	Space-Charge
Acceleration	Defocusing
Lattice	Pervenace

Terms: **Lattice** **Pervenace** **Emittance**

The "acceleration schedule" specifies both  $\gamma_b \beta_b$  and  $\lambda$  then the equations are integrated with:

$$\begin{aligned} \gamma_b \beta_b \varepsilon_x &= \text{const} \\ \gamma_b \beta_b \varepsilon_y &= \text{const} \end{aligned}$$

normalized emittance conservation  
specified pervenace

$$Q = \frac{q\lambda}{2\pi\epsilon_0 r_p^3 \beta_b^2 c^2}$$

**Reminder:** It was shown for a coasting beam that the envelope equations remain valid for elliptic charge densities suggesting more general validity [Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971), J.J. Barnard, [Intro. Lectures](#)]

For any beam with **elliptic symmetry** charge density in each transverse slice:

Based on:

$$\langle x \frac{\partial \phi}{\partial x} \rangle_{\perp} = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

see J.J. Barnard, [Intro. Lectures](#)

the KV envelope equations

$$\begin{aligned} r''_x(s) + \kappa_x(s)r'_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2(s)}{r_x^3(s)} &= 0 \\ r''_y(s) + \kappa_y(s)r'_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2(s)}{r_y^3(s)} &= 0 \end{aligned}$$

remain valid when (averages taken with the full distribution):

$$\begin{aligned} Q &= \frac{q\lambda}{2\pi\epsilon_0 n \gamma_b^3 \beta_b^2 c^2} = \text{const} & \lambda &= q \int d^2x_{\perp} \rho = \text{const} \\ r_x &= 2\langle x^2 \rangle_{\perp}^{1/2} & \varepsilon_x &= 4[\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}^2]^{1/2} \\ r_y &= 2\langle y^2 \rangle_{\perp}^{1/2} & \varepsilon_y &= 4[\langle y^2 \rangle_{\perp} \langle y'^2 \rangle_{\perp} - \langle yy' \rangle_{\perp}^2]^{1/2} \end{aligned}$$

♦ Evolution changes often small in  $\varepsilon_x$ ,  $\varepsilon_y$   
 [SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 33]

## Properties of Envelope Equation Terms:

Applied Focusing:  $\kappa_x r'_x$ ,  $\kappa_y r'_y$  and Acceleration:  $\frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r'_x$ ,  $\frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r'_y$

- ♦ Analogous to single particle orbit terms
- ♦ Contributions to beam envelope essentially the same as in single particle case
- ♦ Have strong  $s$  dependence, can be both focusing and defocusing
  - Act only in focusing elements and acceleration gaps

Perveance:  $\frac{2Q}{r_x + r_y}$

- ♦ Acts continuously in  $s$ , always defocusing
- ♦ Becomes stronger (relative to other terms) when the beam expands in cross-sectional area

Emissittance:  $\frac{\varepsilon_x^2}{r_x^3}$

- ♦ Acts continuously in  $s$ , always defocusing
- ♦ Becomes stronger (relative to other terms) when the beam becomes small in cross-sectional area
- ♦ Scaling makes clear why it is necessary to inhibit emittance growth for applications where small spots are desired on target

[SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 34]

## S5: Matched Envelope Solution:

Neglect acceleration ( $\gamma_b \beta_b = \text{const}$ ) or use transformed variables:

$$\begin{aligned} r''_x(s) + \kappa_x(s)r'_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} &= 0 \\ r''_y(s) + \kappa_y(s)r'_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} &= 0 \end{aligned}$$

Matching involves finding specific initial conditions for the envelope to have the periodicity of the lattice:

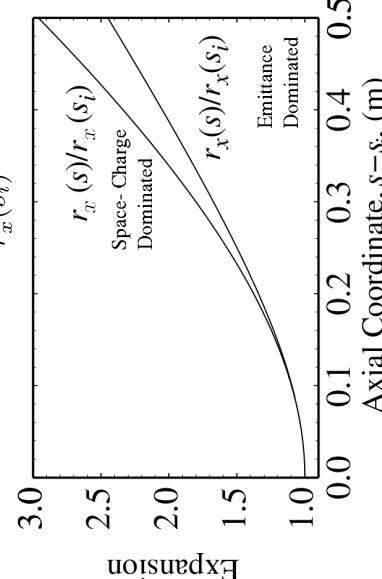
Find Values of:

$$\begin{array}{c} r_x(s_i) \quad r'_x(s_i) \\ r_y(s_i) \quad r'_y(s_i) \end{array}$$

- Such That:
  - $r_x(s_i + L_p) = r_x(s_i)$
  - $r'_x(s_i + L_p) = r'_x(s_i)$
  - $r_y(s_i + L_p) = r_y(s_i)$
  - $r'_y(s_i + L_p) = r'_y(s_i)$
- ♦ Typically constructed with numerical root finding from estimated/guessed values
  - Can be surprisingly difficult for complicated lattices and/or strong space-charge
  - ♦ Iterative technique developed to numerically calculate without root finding
    - [S.M. Lund, S. Chilton and E.P. Lee, PRSTAB 9, 064201 (2006)]

[SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 35]

[SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 36]



See PRSTAB article:  
 solution is analytical in  
 bounding limits shown

Parameters are chosen such  
 that initial defocusing  
 forces in two limits are  
 equal

As the beam expands, the pervenance term will eventually dominate:  
 [see: S.M. Lund and B. Bukh, PRSTAB 7, 024801 (2004)]

Free expansion ( $\kappa_x = \kappa_y = 0$ )

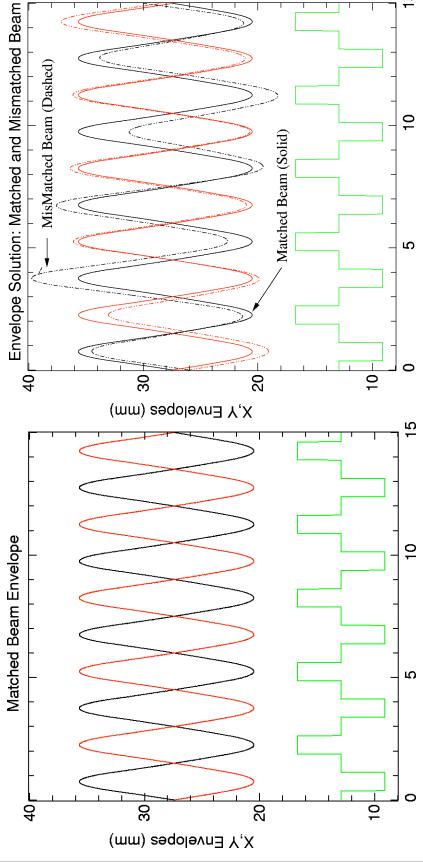
Cases:

$$\begin{array}{ll} \text{Space-Charge Dominated: } \varepsilon_x = 0 & Q \\ \text{Emittance Dominated: } Q = 0 & \frac{Q}{r_x(s_i)} = \frac{\varepsilon_x^2}{r_x^3(s_i)} \\ Q = \frac{\varepsilon_x^2}{r_x^2(s_i)} = 10^{-3} & \end{array}$$

See PRSTAB article:  
 solution is analytical in  
 bounding limits shown

## Typical Matched vs Mismatched solution for FODO channel:

### Matched

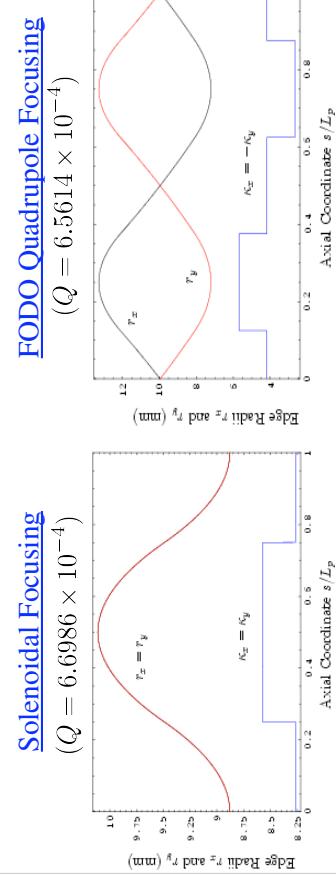


The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport  
 ♦ Matching tends to exploit optics most efficiently to maintain confinement

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 37

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

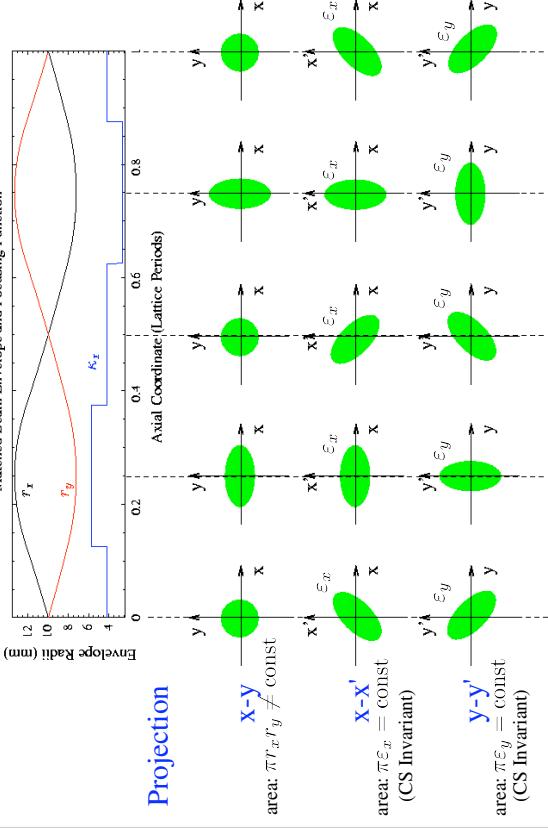
$$\begin{aligned} r_x(s + L_p) &= r_x(s) \\ r_y(s + L_p) &= r_y(s) \\ \varepsilon_x &= \varepsilon_y \\ \sigma/\sigma_0 &= 0.2 \end{aligned}$$



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 38

Symmetries of a matched beam are interpreted in terms of a local rms equivalent KV beam and moments/projections of the KV distribution [see: S.M. Lund, lectures on [Transverse Equilibrium Distributions](#)]

projection beam envelope and focusing function



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 39

## S6: Envelope Perturbations:

An extensive review article is available that both reviews/extends many aspects of envelope modes in periodic lattices covered in [S6-S8](#): see S.M. Lund and B. Bulkh, PRSTAB 024801 (2004) [henceforth denoted: PRSTAB Review]

In the envelope equations set:

### Envelope Perturbations:

$$\begin{aligned} r_x(s) &= r_{xm}(s) + \delta r_x(s) \\ r_y(s) &= r_{ym}(s) + \delta r_y(s) \end{aligned}$$

**Matched Envelope Perturbations**

$$\begin{aligned} r_{xm}(s + L_p) &= r_{xm}(s) & r_{xm}(s) > 0 \\ r_{ym}(s + L_p) &= r_{ym}(s) & r_{ym}(s) > 0 \end{aligned}$$

**Mismatched Envelope Perturbations**

Amplitudes defined in terms of  
 producing small envelope perturbations

♦ Driving terms and distribution errors drive envelope perturbations

- Arise from many sources: focusing errors, lost particles, emittance growth, ....

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 40

The matched solution satisfies:

- ♦ Add subscript  $m$  to denote matched to distinguish from other solutions
- $r_x \rightarrow r_{xm}$  For matched beam envelope
- $r_y \rightarrow r_{ym}$  with periodicity of lattice

$$\begin{aligned} r''_{xm}(s) + \kappa_x(s)r_{xm}(s) - \frac{2Q}{r_{xm}(s) + r_{ym}(s)} - \frac{\varepsilon_x^2}{r_{xm}^3(s)} &= 0 \\ r''_{ym}(s) + \kappa_y(s)r_{ym}(s) - \frac{2Q}{r_{xm}(s) + r_{ym}(s)} - \frac{\varepsilon_y^2}{r_{ym}^3(s)} &= 0 \\ r_{xm}(s + L_p) &= r_{xm}(s) & r_{xm}(s) > 0 \\ r_{ym}(s + L_p) &= r_{ym}(s) & r_{ym}(s) > 0 \end{aligned}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 41

### Matrix Form of the Linearized Perturbed Envelope Equations:

$$\frac{d}{ds} \delta\mathbf{R} + \mathbf{K} \cdot \delta\mathbf{R} = \delta\mathbf{P}$$

$$\begin{aligned} \delta\mathbf{R} &\equiv \begin{pmatrix} \delta r_x \\ \delta r'_x \\ \delta r_y \end{pmatrix} & \text{Coordinate vector} \\ \mathbf{K} &\equiv \begin{pmatrix} 0 & -1 & 0 \\ k_{xm} & 0 & k_{0m} \\ 0 & 0 & -1 \\ k_{0m} & 0 & k_{ym} \end{pmatrix} & \begin{array}{l} \text{Coefficient matrix} \\ k_{0m} = \frac{2Q}{(r_{xm} + r_{ym})^2} \\ k_{jm} = \kappa_j + 3\frac{\varepsilon_j^2}{r_{jm}^4} + k_{0m} \end{array} & \begin{array}{l} \text{Has periodicity} \\ \text{of the lattice period} \end{array} \\ \delta\mathbf{P} &\equiv \begin{pmatrix} 0 \\ -\delta\kappa_x + 2\frac{\delta Q}{r_{xm} + r_{ym}} + 2\frac{\varepsilon_x \delta\varepsilon_x}{r_{xm}^3} \\ 0 \\ -\delta\kappa_y + 2\frac{\delta Q}{r_{xm} + r_{ym}} + 2\frac{\varepsilon_y \delta\varepsilon_y}{r_{ym}^3} \end{pmatrix} & & \text{Driving perturbation vector} \end{aligned}$$

Expand solution into homogeneous and particular parts:

$$\begin{aligned} \delta\mathbf{R} &= \delta\mathbf{R}_h + \delta\mathbf{R}_p & \delta\mathbf{R}_h &= \text{homogeneous solution} \\ \delta\mathbf{R}_p &= \text{particular solution} & \delta\mathbf{R}_p &= \frac{d}{ds} \delta\mathbf{R}_p + \mathbf{K} \cdot \delta\mathbf{R}_p = \delta\mathbf{P} \end{aligned}$$

### Linearized Perturbed Envelope Equations:

- ♦ Neglect all terms of order  $\delta^2$  and higher:  $(\delta r_x)^2, \delta r_x \delta r_y, \delta Q \delta r_x, \dots$

$$\begin{aligned} \delta r''_x + \kappa_x \delta r_x + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4} \delta r_x \\ = -r_{xm} \delta \kappa_x + \frac{2}{r_{xm} + r_{ym}} \delta Q + \frac{2\varepsilon_x^2}{r_{xm}^3} \delta \varepsilon_x \\ \delta r''_y + \kappa_y \delta r_y + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4} \delta r_y \\ = -r_{ym} \delta \kappa_y + \frac{2}{r_{xm} + r_{ym}} \delta Q + \frac{2\varepsilon_y^2}{r_{ym}^3} \delta \varepsilon_y \end{aligned}$$

### Homogeneous Equations:

- ♦ Linearized envelope equations with driving terms set to zero

$$\begin{aligned} \delta r''_x + \kappa_x \delta r_x + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_x^2}{r_{xm}^4} \delta r_x &= 0 \\ \delta r''_y + \kappa_y \delta r_y + \frac{2Q}{(r_{xm} + r_{ym})^2} (\delta r_x + \delta r_y) + \frac{3\varepsilon_y^2}{r_{ym}^4} \delta r_y &= 0 \end{aligned}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 42

### Homogeneous Solution: Normal Modes

- ♦ Describes normal mode oscillations
- ♦ Original analysis by Struckmeier and Reiser [Part. Accel. **14**, 227 (1984)]

### Particular Solution: Driven Modes

- ♦ Describes action of driving terms
- ♦ Characterize in terms of projections on homogeneous response (on normal modes)
- ♦ Homogeneous solution expressible as a map:

$$\begin{aligned} \delta\mathbf{R}(s) &= \mathbf{M}_e(s|s_i) \cdot \delta\mathbf{R}(s_i) \\ \delta\mathbf{R}(s) &= (\delta r_x, \delta r'_x, \delta r_y, \delta r'_y) \\ \mathbf{M}_e(s|s_i) &= 4 \times 4 \text{ transfer map} \end{aligned}$$

Now 4x4 system, but analogous to the 2x2 analysis of Hill's equation via transfer matrices: see S.M. Lund lectures on [Transverse Particle Equations](#)

Eigenvalues and eigenvectors of map through one period characterize normal modes and stability properties:

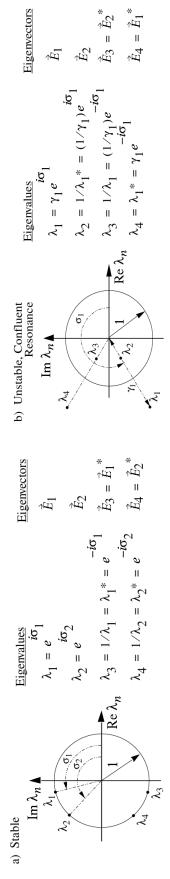
$$\mathbf{M}_e(s_i + L_p|s_i) \cdot \mathbf{E}_n(s_i) = \lambda_n \mathbf{E}_n(s_i)$$

### Stability

$$\begin{aligned} \delta\mathbf{R}(s_i) &= \sum_{n=1}^4 \alpha_n \mathbf{E}_n(s_i) \\ \alpha_n &= \text{const (complex)} \end{aligned}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 44

## Eigenvalue/Eigenvector Symmetry Classes:



**Symmetry classes of eigenvalues/eigenvectors:**

◆ Determine normal mode symmetries

◆ See A. Dragt, Lectures on Nonlinear Orbit Dynamics, in Physics of High Energy Particle Accelerators, (AIP Conf. Proc. No. 87, 1982, p. 147)

◆ Envelope mode symmetries discussed fully in PRSTAB review

◆ Caution: Textbook by Reiser makes errors in mode symmetries and mislabels/identifies dispersion characteristics

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 45

## Pure mode launching conditions:

Launching conditions for distinct normal modes corresponding to the eigenvalue classes illustrated:

$$\begin{aligned} A\ell &= \text{mode amplitude (real)} \\ \psi_\ell &= \text{mode launch phase (real)} \end{aligned}$$

Case	Mode	Launching Condition	Lattice Period Advance
(a) Stable	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1} \mathbf{E}_1 + \mathbf{C}_C$	$\mathbf{M}_e \delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1 + \sigma_1)$
	2 - Stable Osc.	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2} \mathbf{E}_2 + \mathbf{C}_C$	$\mathbf{M}_e \delta\mathbf{R}_2(\psi_2) = \delta\mathbf{R}_2(\psi_2 + \sigma_2)$
(b) Unstable	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1} \mathbf{E}_1 + \mathbf{C}_C$	$\mathbf{M}_e \delta\mathbf{R}_1(\psi_1) = \gamma_1 \delta\mathbf{R}_1(\psi_1 + \sigma_1)$
Confluent Res.	2 - Exp. Damping	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2} \mathbf{E}_2 + \mathbf{C}_C$	$\mathbf{M}_e \delta\mathbf{R}_2(\psi_2) = (1/\gamma_1) \delta\mathbf{R}_2(\psi_2 + \sigma_1)$
(c) Unstable	1 - Stable Osc.	$\delta\mathbf{R}_1 = A_1 e^{i\psi_1} \mathbf{E}_1 + \mathbf{C}_C$	$\mathbf{M}_e \delta\mathbf{R}_1(\psi_1) = \delta\mathbf{R}_1(\psi_1 + \sigma_1)$
Lattice Res.	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2 e^{i\psi_2} \mathbf{E}_2 + \mathbf{C}_C$	$\mathbf{M}_e \delta\mathbf{R}_2 = -\gamma_2 \delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3 \mathbf{E}_4$	$\mathbf{M}_e \delta\mathbf{R}_3 = -(1/\gamma_2) \delta\mathbf{R}_3$
(d) Unstable	1 - Exp. Growth	$\delta\mathbf{R}_1 = A_1 \mathbf{E}_1$	$\mathbf{M}_e \delta\mathbf{R}_1 = -\gamma_1 \delta\mathbf{R}_1$
Double Lattice Resonance	2 - Exp. Growth	$\delta\mathbf{R}_2 = A_2 \mathbf{E}_2$	$\mathbf{M}_e \delta\mathbf{R}_2 = -\gamma_2 \delta\mathbf{R}_2$
	3 - Exp. Damping	$\delta\mathbf{R}_3 = A_3 \mathbf{E}_3$	$\mathbf{M}_e \delta\mathbf{R}_3 = -(1/\gamma_1) \delta\mathbf{R}_3$
	4 - Exp. Damping	$\delta\mathbf{R}_4 = A_4 \mathbf{E}_4$	$\mathbf{M}_e \delta\mathbf{R}_4 = -(1/\gamma_2) \delta\mathbf{R}_4$

$$\delta\mathbf{R}_\ell \equiv \delta\mathbf{R}_\ell(s_i) \quad \mathbf{E}_\ell \equiv \mathbf{E}_\ell(s_i) \quad \mathbf{M}_e \equiv \mathbf{M}_e(s_i + L_p | s_i)$$

$$\delta\mathbf{R}(s) = \begin{cases} A_1 [\mathbf{E}_1(s)e^{i\psi_1(s)} + \mathbf{E}_1^*(s)e^{-i\psi_1(s)}] + A_2 [\mathbf{E}_2(s)e^{-i\psi_2(s)} + \mathbf{E}_2^*(s)e^{-i\psi_2(s)}], & \text{cases (a) and (b)} \\ A_1 [\mathbf{E}_1(s)e^{i\psi_1(s)} + \mathbf{E}_1^*(s)e^{-i\psi_1(s)}] + A_2 [\mathbf{E}_2(s)e^{-i\psi_2(s)} + \mathbf{E}_2^*(s)e^{-i\psi_2(s)}], & \text{case (c)} \\ A_1 \mathbf{E}_1(s) + A_2 \mathbf{E}_2(s) + A_3 \mathbf{E}_3(s) + A_4 \mathbf{E}_4(s), & \text{case (d)} \end{cases}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 46

## Decoupled Modes

In a continuous or periodic solenoidal focusing channel

$$\kappa_x(s) = \kappa_y(s) = \kappa(s)$$

with a round matched-beam solution

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon = \text{const}$$

$r_{xm}(s) = r_{ym}(s) \equiv r_m(s)$  envelope perturbations are simply decoupled with:

$$\text{Breathing Mode: } \delta r_+ \equiv \frac{\delta r_x + \delta r_y}{2} \quad \text{Quadrupole Mode: } \delta r_- \equiv \frac{\delta r_x - \delta r_y}{2}$$

The resulting decoupled envelope equations are:

$$\begin{aligned} \text{Breathing Mode:} \quad & \delta r''_+ + \kappa \delta r_+ + \frac{2Q}{r_m^2} \delta r_+ + \frac{3\varepsilon^2}{r_m^4} \delta r_+ = -r_m \left( \frac{\delta \kappa_x + \delta \kappa_y}{2} \right) + \frac{1}{r_m} \delta Q + \frac{2\varepsilon^2}{r_m^3} \left( \frac{\delta \varepsilon_x + \delta \varepsilon_y}{2} \right) \\ \text{Quadrupole Mode:} \quad & \delta r''_- + \kappa \delta r_- + \frac{3\varepsilon^2}{r_m^4} \delta r_- = -r_m \left( \frac{\delta \kappa_x - \delta \kappa_y}{2} \right) + \frac{2\varepsilon^2}{r_m^3} \left( \frac{\delta \varepsilon_x - \delta \varepsilon_y}{2} \right) \end{aligned}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 47

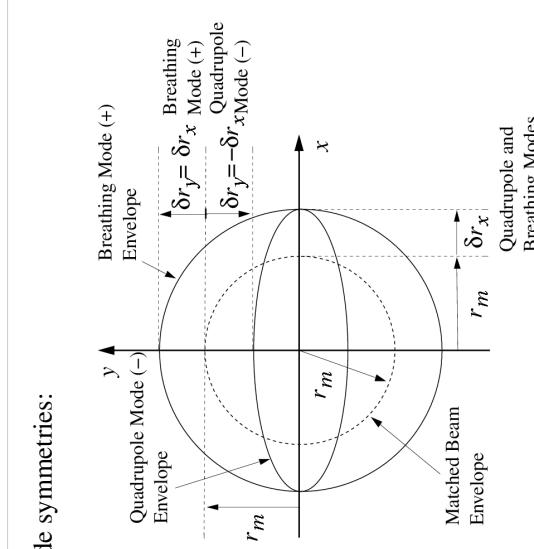
Graphical interpretation of mode symmetries:

Breathing Mode:

$$\delta r^+ = \frac{\delta r_x + \delta r_y}{2}$$

Quadrupole Mode:

$$\delta r^- = \frac{\delta r_x - \delta r_y}{2}$$



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 48

## Decoupled Mode Properties:

Space charge terms  $\sim Q$  only directly expressed in equation for  $\delta r_+(s)$

- ♦ Indirectly present in both equations from matched envelope  $r_m(s)$

### Homogeneous Solution:

- ♦ Restoring term for  $\delta r_+(s)$  larger than for  $\delta r_-(s)$
- Breathing mode should oscillate faster than the quadrupole mode

### Particular Solution:

- ♦ Misbalances in focusing and emittance driving terms can project onto either mode

- nonzero perturbed  $\kappa_x(s) + \kappa_y(s)$  and  $\varepsilon_x(s) + \varepsilon_y(s)$  project onto breathing mode
- nonzero perturbed  $\kappa_x(s) - \kappa_y(s)$  and  $\varepsilon_x(s) - \varepsilon_y(s)$  project onto quadrupole mode
- ♦ Perveance driving perturbations project *only* on breathing mode

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 49

## Previous symmetry classes greatly reduce for decoupled modes:

Previous homogeneous 4x4 solution map:

$$\begin{aligned}\delta\mathbf{R}(s) &= \mathbf{M}_e(s|s_i) \cdot \delta\mathbf{R}(s_i) \\ \delta\mathbf{R}'(s) &= (\delta r_x, \delta r'_x, \delta r_y, \delta r'_y)\end{aligned}$$

$\mathbf{M}_e(s|s_i) = 4 \times 4$  transfer map

reduces to two independent 2x2 maps with greatly simplified symmetries:

$$\delta\mathbf{R} \equiv (\delta r_+, \delta r'_+, \delta r_-, \delta r'_-)$$

$$\mathbf{M}_e(s_i + L_p|s_i) = \begin{bmatrix} \mathbf{M}_+(s_i + L_p|s_i) & 0 \\ 0 & \mathbf{M}_-(s_i + L_p|s_i) \end{bmatrix}$$

with corresponding eigenvalue problems:

$$\mathbf{M}_{\pm}(s_i + L_p|s_i) \cdot \mathbf{E}_n(s_i) = \lambda_{\pm} \mathbf{E}_n(s_i)$$

Many familiar results from analysis of Hills equation (see: S.M. Lund lectures on **Transverse Particle Equations**) can be immediately applied to the decoupled case, for example:

$$\frac{1}{2} |\text{Tr } \mathbf{M}_{\pm}(s_i + L_p|s_i)| < 1 \quad \longrightarrow \quad \text{mode stability}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 50

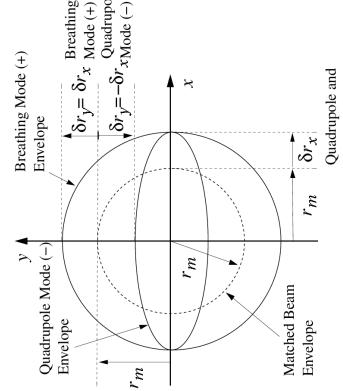
## Eigenvalue symmetries and launching conditions simplify for decoupled modes

### Eigenvalue Symmetry 1:

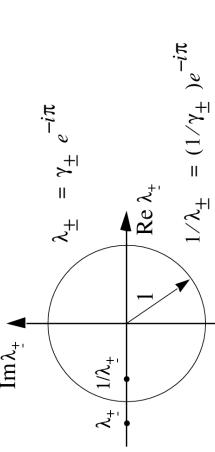
Stable

$$\lambda_{\pm} = e^{i\sigma_{\pm}} \quad \lambda_{\pm}^* = 1/\lambda_{\pm} = e^{-i\sigma_{\pm}}$$

### Launching Condition / Projections



### Eigenvalue Symmetry 2: Unstable, Lattice Resonance



## General Mode Limits

Using phase-amplitude analysis can show for any linear focusing lattice:

- 1) Phase advance of any normal mode satisfies the zero space-charge limit:

$$\lim_{Q \rightarrow 0} \sigma_{\ell} = 2\sigma_0$$

- 2) Pure normal modes evolve with a quadratic phase-space

(Courant-Snyder) invariant in the normal coordinates of the mode

Simply expressed for decoupled modes with  $\kappa_x = \kappa_y$ ,  $\varepsilon_x = \varepsilon_y$

$$\left[ \frac{\delta r_{\pm}(s)}{w_{\pm}(s)} \right]^2 + [w_{\pm}(s)\delta r_{\pm}(s) - w_{\pm}(s)\delta r'_{\pm}(s)]^2 = \text{const}$$

where

$$\begin{aligned}w_+'' + \kappa w_+ + \frac{2Q}{r_m^2} w_+ + \frac{3\varepsilon^2}{r_m^4} w_+ - \frac{1}{w_+^3} &= 0 \\ w_-'' + \kappa w_- + \frac{2Q}{r_m^2} w_- + \frac{3\varepsilon^2}{r_m^4} w_- - \frac{1}{w_-^3} &= 0 \\ w_{\pm}(s + L_p) &= w_{\pm}(s)\end{aligned}$$

Analogous results for coupled modes [See Edwards and Teng, IEEE Trans Nuc. Sci. **20**, 885 (1973)]

- ♦ More complex expression due to coupling

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 51

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 52

## S7: Envelope Modes in Continuous Focusing

**Focusing:**  $\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2 = \text{const}$

**Matched beam:**

symmetric beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$

$$r_{xm}(s) = r_{ym}(s) = r_m = \text{const}$$

matched envelope:

$$k_{\beta 0}^2 r_m - \frac{Q}{r_m} - \frac{\varepsilon^2}{r_m^3} = 0$$

$$\text{depressed phase advance: } \sigma = \sqrt{\sigma_0^2 - \frac{Q}{(r_m/L_p)^2}} = \frac{\varepsilon L_p}{r_m^2}$$

one parameter needed for scaled solution:

$$\frac{k_{\beta 0}^2 \varepsilon^2}{Q^2} = \frac{\sigma_0^2 \varepsilon^2}{Q^2 L_p^2} = \frac{(\sigma/\sigma_0)^2}{[1 - (\sigma/\sigma_0)^2]^2}$$

**Decoupled Modes:**

$$\delta r_{\pm}(s) = \frac{\delta r_x(s) \pm \delta r_y(s)}{2}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 53

**Envelope equations of motion become:**

$$\begin{aligned} I_p^2 \frac{d^2}{ds^2} \left( \frac{\delta r_+}{r_m} \right) + \sigma_+^2 \left( \frac{\delta r_+}{r_m} \right) &= -\frac{\sigma_0^2}{2} \left( \frac{\delta \kappa_x}{k_{\beta 0}^2} + \frac{\delta \kappa_y}{k_{\beta 0}^2} \right) + (\sigma_0^2 - \sigma^2) \frac{\delta Q}{Q} + \sigma^2 \left( \frac{\delta \varepsilon_x}{\varepsilon} + \frac{\delta \varepsilon_y}{\varepsilon} \right) \\ I_p^2 \frac{d^2}{ds^2} \left( \frac{\delta r_-}{r_m} \right) + \sigma_-^2 \left( \frac{\delta r_-}{r_m} \right) &= -\frac{\sigma_0^2}{2} \left( \frac{\delta \kappa_x}{k_{\beta 0}^2} - \frac{\delta \kappa_y}{k_{\beta 0}^2} \right) + \sigma^2 \left( \frac{\delta \varepsilon_x}{\varepsilon} - \frac{\delta \varepsilon_y}{\varepsilon} \right) \end{aligned}$$

$$\sigma_+ \equiv \sqrt{2\sigma_0^2 + 2\sigma^2} \quad \text{"breathing" mode phase advance}$$

$$\sigma_- \equiv \sqrt{\sigma_0^2 + 3\sigma^2} \quad \text{"quadrupole" mode phase advance}$$

**Homogeneous equations for normal modes:**

$$\frac{d^2}{ds^2} \delta r_{\pm} + \left( \frac{\sigma_{\pm}}{L_p} \right)^2 \delta r_{\pm} = 0$$

♦ Simple harmonic oscillator equation

**Homogeneous Solution (normal modes):**

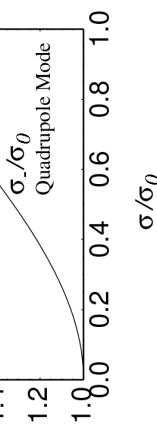
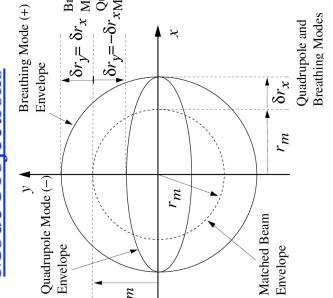
$$\delta r_{\pm}(s) = \delta r_{\pm}(s_i) \cos \left( \sigma_{\pm} \frac{s - s_i}{L_p} \right) + \frac{\delta r'_{\pm}(s_i)}{\sigma_{\pm}/L_p} \sin \left( \sigma_{\pm} \frac{s - s_i}{L_p} \right)$$

$$\delta r_{\pm}(s_i), \quad \delta r'_{\pm}(s_i) \quad \text{mode initial conditions}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 54

**Properties of continuous focusing homogeneous solution: Normal Modes**

Mode Projections



$$\text{Breathing Mode: } \delta r_+ \equiv \frac{\delta r_x + \delta r_y}{2}$$

$$\text{Quadrupole Mode: } \delta r_- \equiv \frac{\delta r_x - \delta r_y}{2}$$

Green's function solution is *fully general*. Insight gained from simplified solutions for specific classes of driving perturbations:

♦ Adiabatic

covered here

Ramped

covered in PRSTAB Review article

♦ Sudden

♦ Harmonic

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 55

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 56

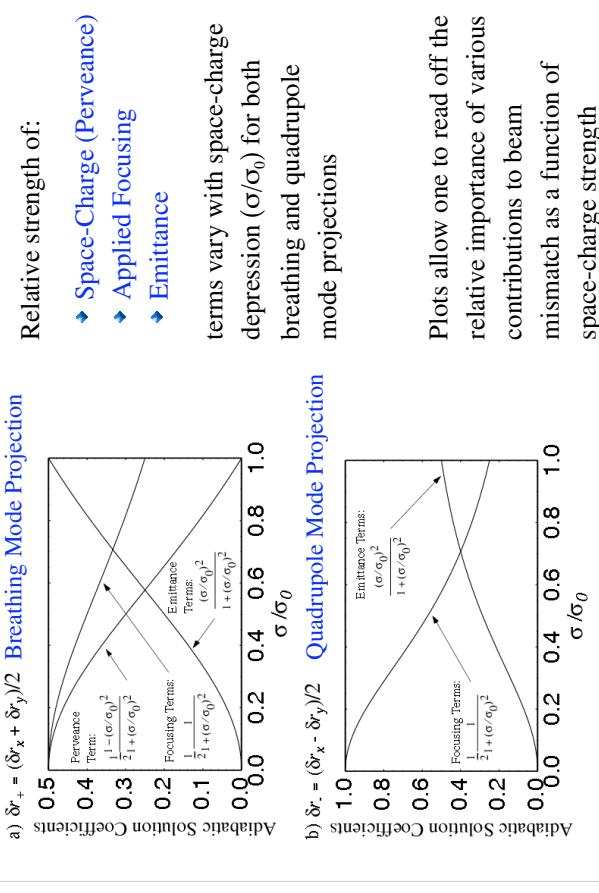
## Continuous Focusing – adiabatic particular solution

For driving perturbations  $\delta p_+(s)$  and  $\delta p_-(s)$  slow on quadrupole mode (slower mode) wavelength  $\sim 2\pi L_p/\sigma_-$  the solution is:

$$\begin{aligned} \frac{\delta r_+(s)}{r_m} &= \frac{\delta p_+(s)}{\sigma_+^2} && \text{Focusing} && \text{Pervance} \\ &= -\left[\frac{1}{2}\frac{1}{1+(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} + \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right) + \left[\frac{1}{2}\frac{1-(\sigma/\sigma_0)^2}{1+(\sigma/\sigma_0)^2}\right]\frac{\delta Q(s)}{Q} \\ &\quad + \left[\frac{(\sigma/\sigma_0)^2}{1+(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\varepsilon_x(s)}{\varepsilon} + \frac{\delta\varepsilon_y(s)}{\varepsilon}\right), && \text{Emissance} && \text{Coefficients of adiabatic terms in square brackets "[ ]"]} \\ \frac{\delta r_-(s)}{r_m} &= \frac{\delta p_-(s)}{\sigma_-^2} && \text{Focusing} && \text{Emissance} \\ &= -\left[\frac{1}{1+3(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\kappa_x(s)}{k_{\beta 0}^2} - \frac{\delta\kappa_y(s)}{k_{\beta 0}^2}\right) \\ &\quad + \left[\frac{2(\sigma/\sigma_0)^2}{1+3(\sigma/\sigma_0)^2}\right]\frac{1}{2}\left(\frac{\delta\varepsilon_x(s)}{\varepsilon} - \frac{\delta\varepsilon_y(s)}{\varepsilon}\right). && \text{Emissance} && \end{aligned}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 57

## Continuous Focusing – adiabatic solution coefficients



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 58

## Continuous Focusing – sudden particular solution

For step function driving perturbations of form:

$$\delta p_{\pm}(s) = \widehat{\delta p}_{\pm}\Theta(s - s_p)$$

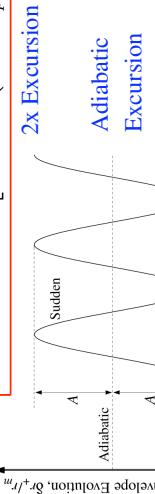
with amplitudes:

$$\widehat{\delta p}_+ = -\frac{\sigma_0^2}{2}\left[\frac{\widehat{\delta\kappa}_x}{k_{\beta 0}^2} + \frac{\widehat{\delta\kappa}_y}{k_{\beta 0}^2}\right] + (\sigma_0^2 - \sigma^2)\frac{\widehat{\delta Q}}{Q} + \sigma^2\left[\frac{\widehat{\delta\varepsilon}_x}{\varepsilon} + \frac{\widehat{\delta\varepsilon}_y}{\varepsilon}\right] = \text{const}$$

$$\widehat{\delta p}_- = -\frac{\sigma_0^2}{2}\left[\frac{\widehat{\delta\kappa}_x}{k_{\beta 0}^2} - \frac{\widehat{\delta\kappa}_y}{k_{\beta 0}^2}\right] + \sigma^2\left[\frac{\widehat{\delta\varepsilon}_x}{\varepsilon} - \frac{\widehat{\delta\varepsilon}_y}{\varepsilon}\right] = \text{const}$$

The solution is given by the substitution in the expression for the adiabatic solution:

$$\delta p_{\pm}(s) \rightarrow \widehat{\delta p}_{\pm}\left[1 - \cos\left(\sigma_{\pm}\frac{s - s_p}{L_p}\right)\right]\Theta(s - s_p)$$



For the same amplitude of total driving perturbations, sudden perturbations result in 2x the envelope excursion that adiabatic perturbations produce.

## S8: Envelope Modes in Periodic Focusing Channels

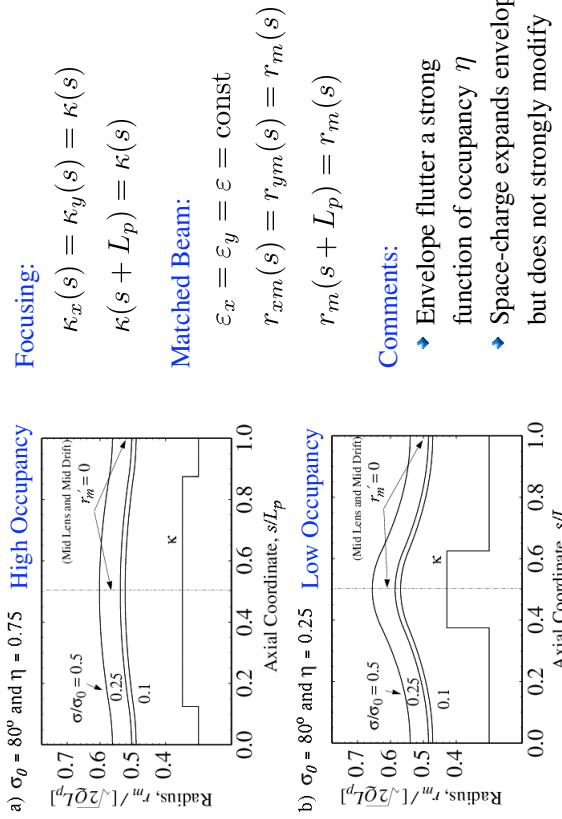
### Overview

- ♦ Space-Charge (Perveance)
- ♦ Applied Focusing
- ♦ Emittance
- ♦ Relative strength of:
- terms with space-charge
- depression ( $\sigma/\sigma_0$ ) for both breathing and quadrupole mode projections
- plots allow one to read off the relative importance of various contributions to beam mismatch as a function of space-charge strength
- ♦ Much more complicated the continuous limit results
- lattice can couple to oscillations and destabilize the system
- broad parametric instability bands can result
- instability bands calculated will exclude wide ranges of parameter space from machine operation
- exclusion region depends on focusing type
- will find that alternating gradient quadrupole focusing tends to have more instability than high occupancy solenoidal focusing due to larger envelope flutter driving stronger, broader instability
- results in this section are calculated numerically and summarized parametrically to illustrate the full range of mode characteristics
- results presented in terms of phase advances and normalized space-charge strength to allow broad applicability
- coupled 4x4 eigenvalue problem and mode symmetries identified in S6 are solved numerically and analytical limits are verified
- more information on results presented can be found in the PRSTAB Review

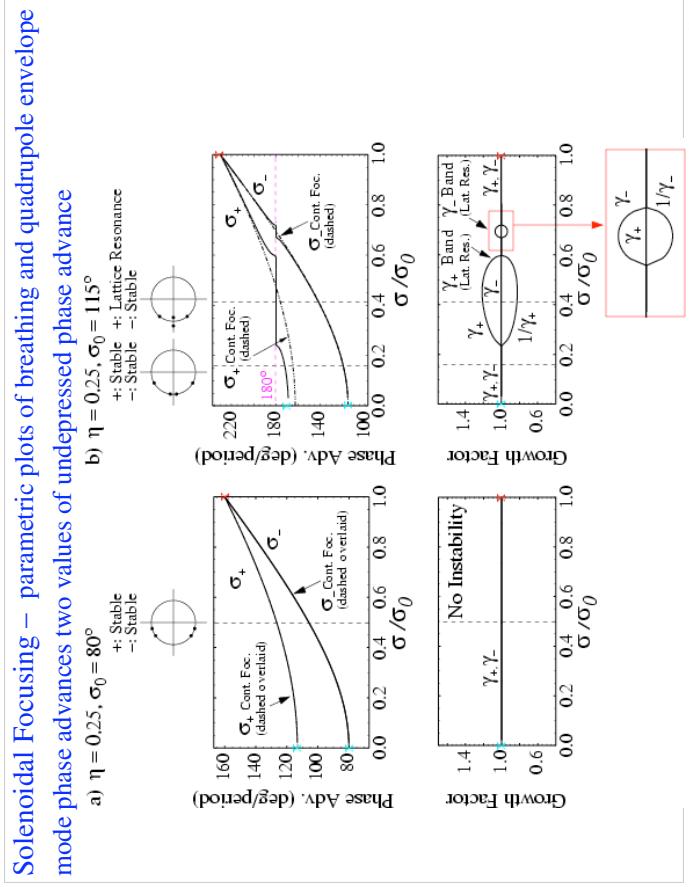
SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 59

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 60

## Solenoidal Focusing – Matched Envelope Solution



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 61



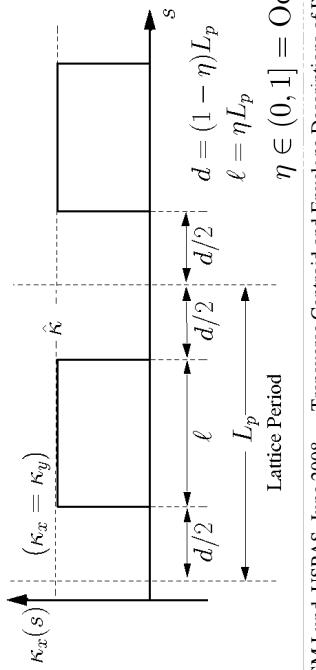
SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 63

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:

♦ See: S.M. Lund, lectures on **Transverse Particle Equations**

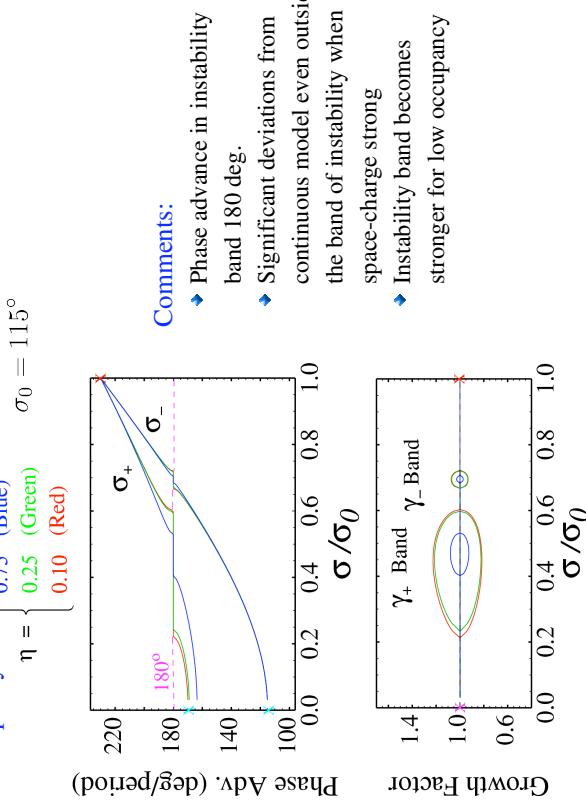
**Solenoidal Focusing - piecewise constant focusing lattice**

$$\cos \sigma_0 = \cos(2\Theta) - \frac{1-\eta}{\eta} \Theta \sin(2\Theta) \quad \Theta \equiv \frac{\sqrt{\kappa} L_p}{2}$$



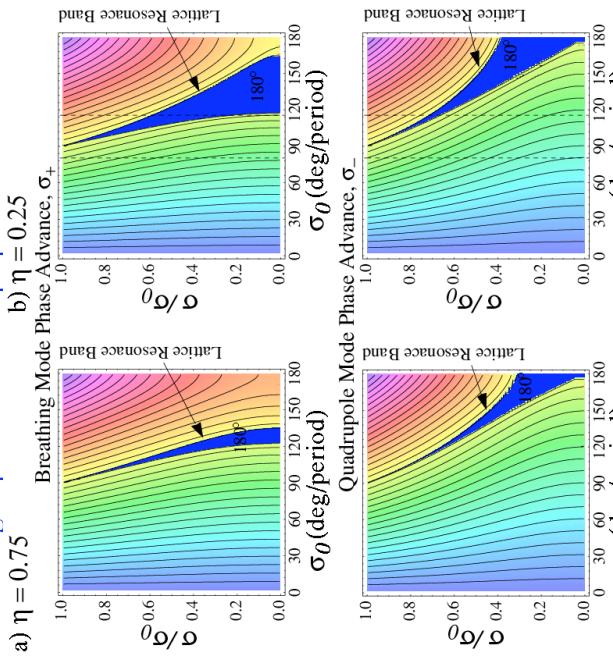
SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 62

**Solenoidal Focusing – mode instability bands become wider and stronger for smaller occupancy**



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 64

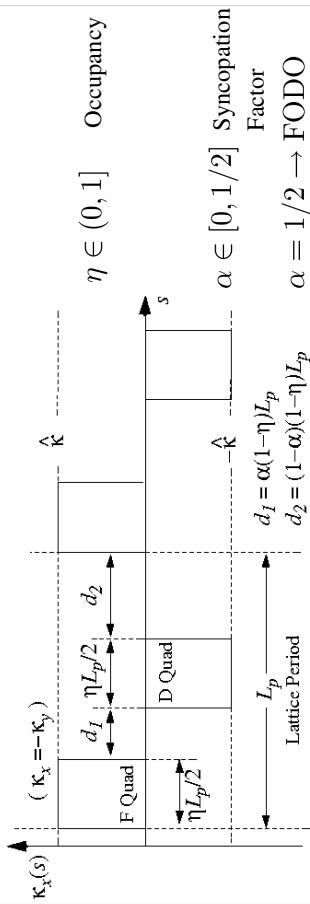
### Solenoidal Focusing – parametric mode properties of band oscillations



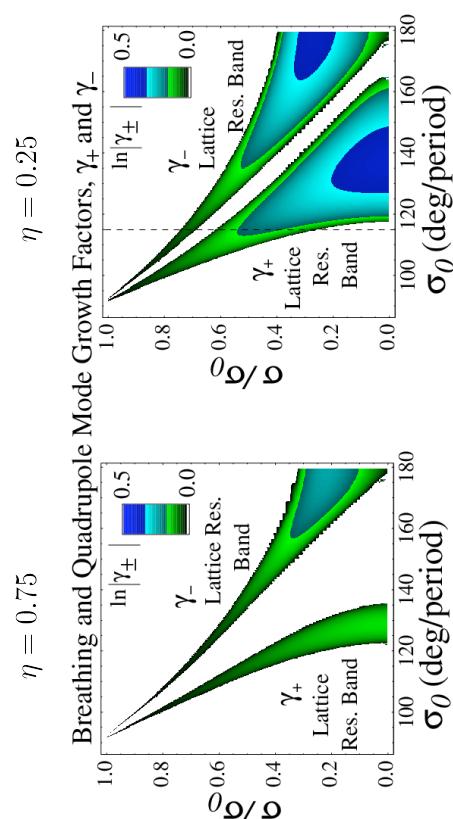
SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 66

Using a transfer matrix approach on undepressed single-particle orbits set the strength of the focusing function for specified undepressed particle phase advance by solving:  
See: S.M. Lund, lectures on [Transverse Particle Equations](#)

$$\cos \sigma_0 = \cos \Theta \cosh \Theta + \frac{1-\eta}{\eta} \theta (\cos \Theta \sinh \Theta - \sin \Theta \cosh \Theta) \\ - 2\alpha(1-\alpha) \frac{(1-\eta)^2}{\eta^2} \Theta^2 \sin \Theta \sinh \Theta$$



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 68



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 65

### Quadrupole Doublet Focusing – Matched Envelope Solution

#### FODO and Syncopated Lattices

a)  $\sigma_0 = 80^\circ$ ,  $\eta = 0.6949$ , and  $\alpha = 1/2$  FODO

Focusing:

$$\kappa_x(s) = -\kappa_y(s) = \kappa(s)$$

$$\kappa(s + L_p) = \kappa(s)$$

Matched Beam:

$$\varepsilon_x = \varepsilon_y = \varepsilon = \text{const}$$

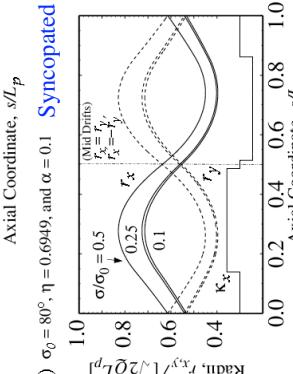
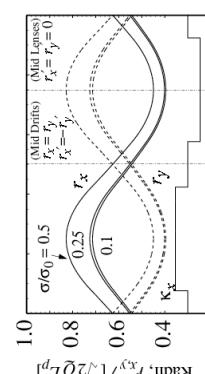
$$r_{xm}(s + L_p) = r_{xm}(s)$$

$$r_{ym}(s + L_p) = r_{ym}(s)$$

Comments:

- Envelope flutter a *weak* function of occupancy  $\eta$
- Syncopation factors  $\alpha \neq 1/2$  reduce envelope symmetry and can drive more instabilities
- Space-charge expands envelope

b)  $\sigma_0 = 80^\circ$ ,  $\eta = 0.6949$ , and  $\alpha = 0.1$  Syncopated



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 67

**Quadrupole Focusing – parametric plots of breathing and quadrupole envelope mode phase advances two values of undepressed phase advance**

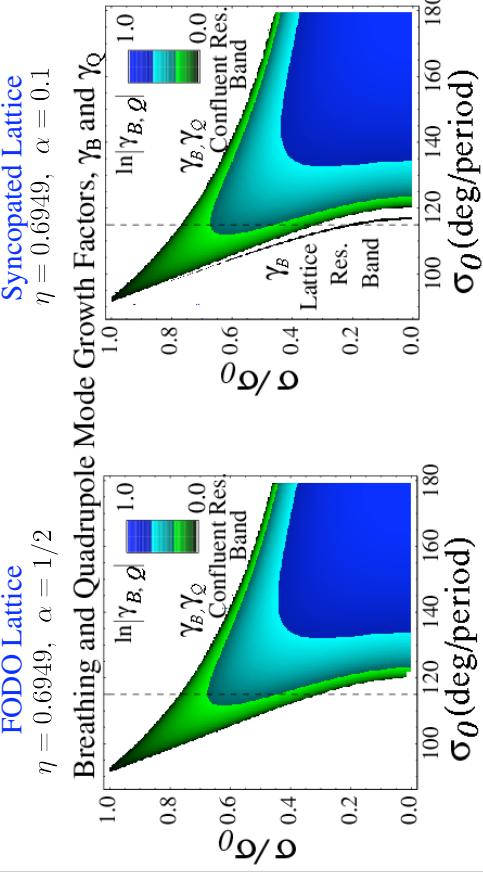
a)  $\eta = 0.6949, \alpha = 0.1, \sigma_0 = 80^\circ$   
b)  $\eta = 0.6949, \alpha = 0.1, \sigma_0 = 115^\circ$

**FODO**

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 69

**Quadrupole Focusing – broad ranges of parametric instability are found for the breathing and quadrupole bands that must be avoided in machine operation**

**FODO Lattice**  
 $\eta = 0.6949, \alpha = 1/2$

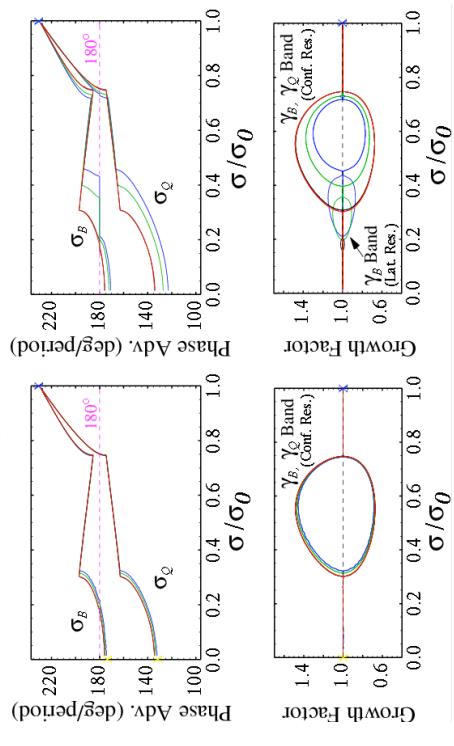


**Quadrupole Focusing – mode instability bands vary little/strongly with occupancy for FODO/syncopated lattices**

a)  $\alpha = 1/2$  (FODO),  $\sigma_0 = 115^\circ$   
b)  $\alpha = 0.1, \sigma_0 = 115^\circ$

**FODO**

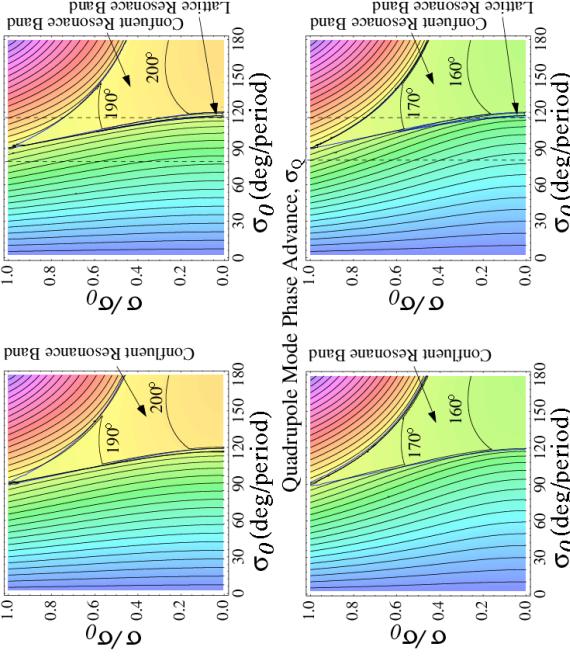
$$\eta = \begin{cases} 0.90 & (\text{Blue}) \\ 0.6949 & (\text{Black}) \\ 0.25 & (\text{Green}) \\ 0.10 & (\text{Red}) \end{cases}$$



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 70

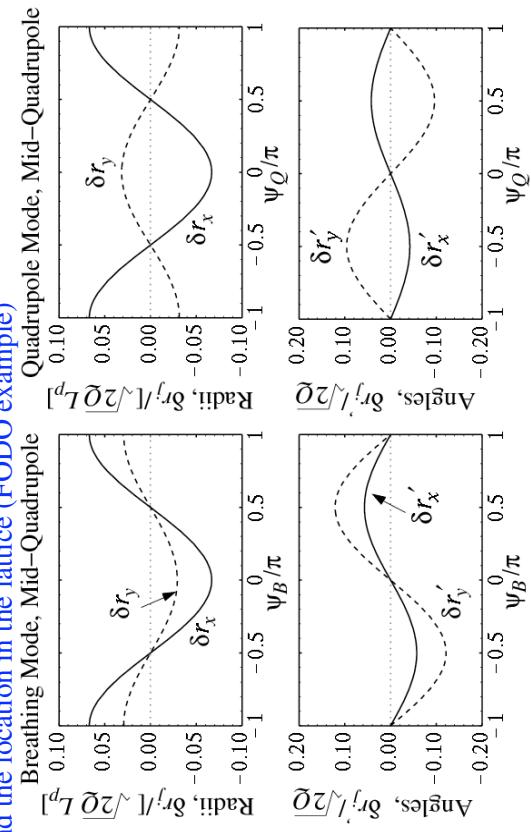
**Quadrupole Focusing – parametric mode properties of band oscillations**

a)  $\eta = 0.6949, \alpha = 1/2$  **FODO**  
b)  $\eta = 0.6949, \alpha = 0.1$  **Syncopated**



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 72

## Quadrupole Focusing – mode structure varies strongly with mode phase and the location in the lattice (FODO example)

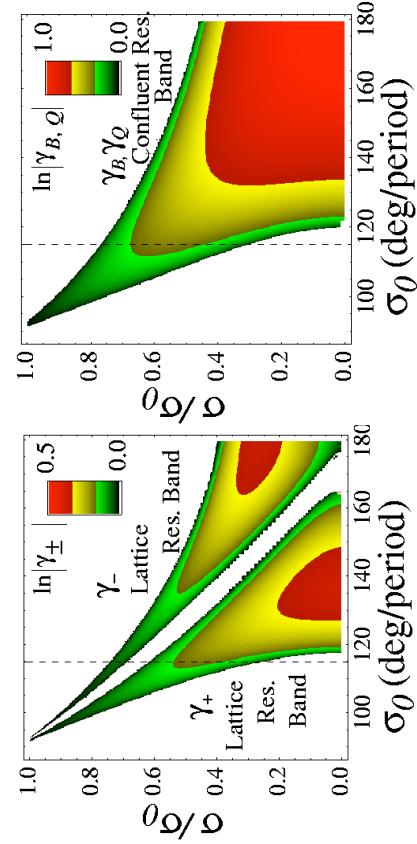


**Summary:** Envelope band instabilities and growth rates for periodic solenoidal and quadrupole doublet focusing lattices

## Envelope Mode Instability Growth Rates

### Solenoid ( $\eta = 0.25$ )

### Quadrupole FODO ( $\eta = 0.70$ )

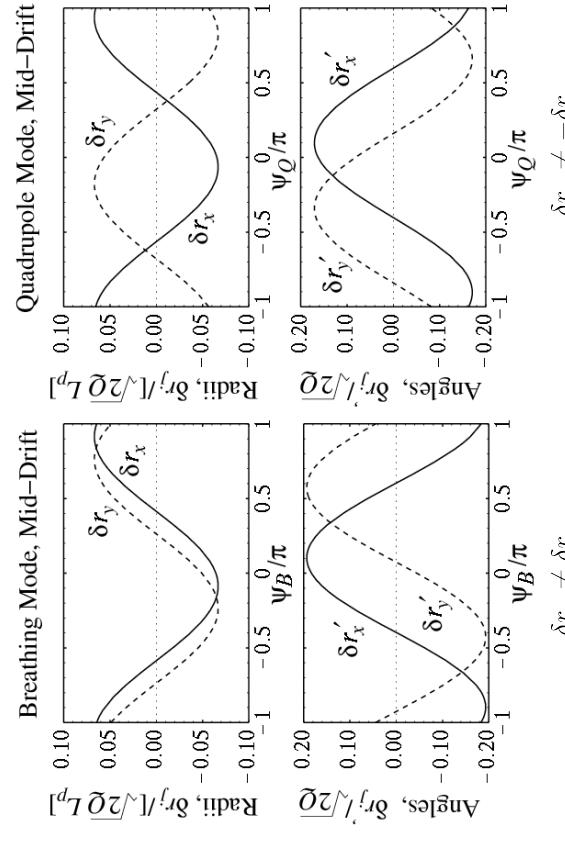


[S.M. Lund and B. Bulth, PRSTAB 024801 (2004)]

## Quadrupole Mode, Mid-Drift



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 73



SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 74

## S9: Transport Limit Scaling Based on Envelope Models

See Handwritten Notes from 2006 USPAS

- Will attempt to convert to slides in future versions of the class
- generally not exact
- breathing symmetry

generally not exact

breathing symmetry

## S8: Centroid and Envelope Descriptions via 1<sup>st</sup> Order Coupled Moment Equations

When constructing centroid and moment models, it can be efficient to simply write moments, differentiate them, and then apply the equation of motion.

Generally, this results in lower order moments coupling to higher order ones and an infinite chain of equations. But the hierarchy can be truncated by:

- ♦ Assuming a fixed functional form of the distribution in terms of moments
- ♦ Neglecting coupling to higher order terms

Resulting first order moment equations can be expressed in terms of a closed set of moments and advanced in  $s$  or  $t$  using simple (ODE based) numerical codes. This approach can prove simpler to include effects where invariants are not easily extracted to reduce the form of the equations (as when solving the KV envelope equations in the usual form).

**Examples of effects that might be more readily analyzed:**

- ♦ See: references at end of notes
- ♦ J.J. Barnard, lecture on Heavy-Ion Fusion and Final Focusing
- ♦ Dispersion in bends
- ♦ Dispersion in quadrupoles
- ♦ Chromatic effects in final focus

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 77

When simplifying the results, if the distribution form is frozen in terms of moments (Example: assume uniform density elliptical beam) then we use constructs like:

$$n = \int d^2x'_\perp f_\perp = n(\mathbf{M})$$

to simplify the resulting equations and express the RHS in terms of elements of  $\mathbf{M}$

1<sup>st</sup> order moments:

$\mathbf{X}_\perp = \langle \mathbf{x}_\perp \rangle_\perp$	Centroid coordinate
$\mathbf{X}'_\perp = \langle \mathbf{x}'_\perp \rangle_\perp$	Centroid angle
+ possible others if more variables. Example	
$\Delta = \langle \frac{\delta p_s}{p_s} \rangle = \langle \delta \rangle$	Centroid off-momentum
:	:

Resulting form of coupled moment equations:

$$\frac{d}{ds} \mathbf{M} = \mathbf{F}(\mathbf{M})$$

- ♦  $\mathbf{M}$  = vector of moments, generally infinite
- ♦  $\mathbf{F}$  = vector function of  $\mathbf{M}$ , generally nonlinear
- ♦ System advanced from a specified initial condition (initial value of  $\mathbf{M}$ )

Transverse moment definition:

$$\langle \dots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \dots f_\perp}{\int d^2x_\perp \int d^2x'_\perp f_\perp}$$

Can be generalized if other variables such as off momentum are included in  $f$

Differentiate moments and apply equations of motion:

$$\frac{d}{ds} \langle \dots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \left[ \frac{d}{ds} \dots \right] f_\perp}{\int d^2x_\perp \int d^2x'_\perp f_\perp} + \text{apply equations of motion to simplify } \frac{d}{ds} \dots$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 78

2<sup>nd</sup> order moments:

It is typically convenient to subtract centroid from higher-order moments

$$\begin{aligned} \tilde{x} &\equiv x - X & \tilde{x}' &\equiv x' - X' \\ \tilde{y} &\equiv y - Y & \tilde{y}' &\equiv y' - Y' \\ \tilde{\delta} &\equiv \delta - \Delta \end{aligned}$$

x-moments	y-moments	x-y cross moments	dispersive moments
$\langle \tilde{x}^2 \rangle_\perp$	$\langle \tilde{y}^2 \rangle_\perp$	$\langle \tilde{x}\tilde{y} \rangle_\perp$	$\langle \tilde{x}\tilde{\delta} \rangle$ , $\langle \tilde{y}\tilde{\delta} \rangle$
$\langle \tilde{x}\tilde{x}' \rangle_\perp$	$\langle \tilde{y}\tilde{y}' \rangle_\perp$	$\langle \tilde{x}'\tilde{y} \rangle_\perp$ , $\langle \tilde{x}\tilde{y}' \rangle_\perp$	$\langle \tilde{x}'\tilde{\delta} \rangle$ , $\langle \tilde{y}'\tilde{\delta} \rangle$
$\langle \tilde{x}'^2 \rangle_\perp$	$\langle \tilde{y}'^2 \rangle_\perp$	$\langle \tilde{x}'\tilde{y}' \rangle_\perp$	$\langle \tilde{\delta}^2 \rangle$

3<sup>rd</sup> order moments: Analogous to 2<sup>nd</sup> order case, but more for each order

$$\langle \tilde{x}^3 \rangle_\perp, \langle \tilde{x}^2\tilde{y} \rangle_\perp, \dots$$

Many quantities of physical interest are expressed in transport can then be expressed in terms of moments calculated when the equations are numerically advanced in  $s$  and their evolutions plotted to understand behavior

- ♦ Many quantities of physical interest are expressible in terms of 1st and 2nd order moments

Example moments often projected:

**Statistical beam size:**  
(rms edge measure)

$$\begin{aligned} r_x &= 2\langle \tilde{x}^2 \rangle_{\perp}^{1/2} \\ r_y &= 2\langle \tilde{y}^2 \rangle_{\perp}^{1/2} \end{aligned}$$

Kinetic longitudinal temperature:

(rms measure)

$$T_s = \text{const} \times \langle \tilde{\delta}^2 \rangle$$

## Illustrate approach with the familiar KV model

Truncation assumption: unbunched uniform density elliptical beam in free space

- ♦  $\delta = 0$ , no axial velocity spread
- ♦ All cross moments zero, i.e.  $\langle \tilde{x}\tilde{y} \rangle_{\perp} = 0$

$$\begin{aligned} \frac{d}{ds} \langle x \rangle_{\perp} &= \langle x' \rangle_{\perp} & \frac{d}{ds} \langle x^2 \rangle_{\perp} &= 2\langle xx' \rangle_{\perp} \\ \frac{d}{ds} \langle x' \rangle_{\perp} &= \langle x'' \rangle_{\perp} & \frac{d}{ds} \langle x'^2 \rangle_{\perp} &= 2\langle x'x'' \rangle_{\perp} \\ &\vdots & &\vdots \end{aligned}$$

Use particle equations of motion within beam, neglect images, and simplify

- ♦ Apply equations in S2 with  $\mathbf{E}_{\perp}^i = 0$

$$\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x' + \kappa_x x - \frac{2Q}{(r_x + r_y)r_x} (x - \langle x \rangle_{\perp}) &= 0 \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} y' + \kappa_y y - \frac{2Q}{(r_x + r_y)r_y} (y - \langle y \rangle_{\perp}) &= 0 \end{aligned}$$

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 81

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 82

## Resulting system of 1st and 2nd order moments

**1<sup>st</sup> order moments:**

$$\frac{d}{ds} \begin{bmatrix} \langle x \rangle_{\perp} \\ \langle x' \rangle_{\perp} \\ \langle y \rangle_{\perp} \\ \langle y' \rangle_{\perp} \end{bmatrix} = \begin{bmatrix} \langle x' \rangle_{\perp} \\ -\kappa_x(s)\langle x \rangle_{\perp} \\ \langle y \rangle_{\perp} \\ -\kappa_y(s)\langle y \rangle_{\perp} \end{bmatrix}$$

**2<sup>nd</sup> order moments:**

$$\begin{aligned} \frac{d}{ds} \begin{bmatrix} \langle \tilde{x}^2 \rangle_{\perp} \\ \langle \tilde{x}\tilde{x}' \rangle_{\perp} \\ \langle \tilde{x}'^2 \rangle_{\perp} \\ \langle \tilde{y}^2 \rangle_{\perp} \\ \langle \tilde{y}\tilde{y}' \rangle_{\perp} \\ \langle \tilde{y}'^2 \rangle_{\perp} \end{bmatrix} &= \begin{bmatrix} 2\langle \tilde{x}\tilde{x}' \rangle_{\perp} \\ \langle \tilde{x}^2 \rangle_{\perp} - \kappa_x(s)\langle \tilde{x}^2 \rangle_{\perp} + \frac{Q\langle \tilde{x}'^2 \rangle_{\perp}}{[4\langle \tilde{x}^2 \rangle_{\perp}^{1/2}(\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \\ -2\kappa_x(s)\langle \tilde{x}\tilde{x}' \rangle_{\perp} + \frac{Q\langle \tilde{x}'^2 \rangle_{\perp}}{[4\langle \tilde{x}^2 \rangle_{\perp}^{1/2}(\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \\ 2\langle \tilde{y}\tilde{y}' \rangle_{\perp} \\ \langle \tilde{y}'^2 \rangle_{\perp} - \kappa_y(s)\langle \tilde{y}^2 \rangle_{\perp} + \frac{Q\langle \tilde{y}'^2 \rangle_{\perp}}{[4\langle \tilde{y}^2 \rangle_{\perp}^{1/2}(\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \\ -2\kappa_y(s)\langle \tilde{y}\tilde{y}' \rangle_{\perp} + \frac{Q\langle \tilde{y}'^2 \rangle_{\perp}}{[4\langle \tilde{y}^2 \rangle_{\perp}^{1/2}(\langle \tilde{x}^2 \rangle_{\perp}^{1/2} + \langle \tilde{y}^2 \rangle_{\perp}^{1/2})]} \end{bmatrix} \end{aligned}$$

- ♦ Express 1st and 2nd order moments separately in this case since uncoupled
- ♦ Form truncates due to frozen distribution form: all moments on LHS on RHS
- ♦ Integrate from initial moments values of  $s$  and project out desired quantities

## Using 2<sup>nd</sup> order moment equations we can show that

$$\begin{aligned} \frac{d}{ds} \varepsilon_x^2 &= 0 = \frac{d}{ds} \varepsilon_y^2 \\ \varepsilon_x^2 &= 16 [\langle x^2 \rangle_{\perp} \langle x'^2 \rangle_{\perp} - \langle xx' \rangle_{\perp}^2] = \text{const} \\ \varepsilon_y^2 &= 16 [\langle y^2 \rangle_{\perp} \langle y'^2 \rangle_{\perp} - \langle yy' \rangle_{\perp}^2] = \text{const} \\ \Rightarrow & \end{aligned}$$

Using this, the 2<sup>nd</sup> order moment equations can be equivalently expressed in the standard KV envelope form:

$$\begin{aligned} \frac{dr_x}{ds} &= r'_x ; & \frac{d}{ds} r'_x + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} &= 0 \\ \frac{dr_y}{ds} &= r'_y ; & \frac{d}{ds} r'_y + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} &= 0 \end{aligned}$$

- ♦ Moment form fully consistent with usual KV model ... as it must be
- ♦ Moment form generally easier to put in additional effects that would violate the usual emittance invariants

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 83

SM Lund, USPAS, June 2008 Transverse Centroid and Envelope Descriptions of Beam Evolution 84

Relative advantages of the use of coupled matrix form versus reduced equations can depend on the problem/situation

### Coupled Matrix Equations

$$\frac{d}{ds} \mathbf{M} = \mathbf{F}$$

**M** = Moment Vector  
**F** = Force Vector

$$X'' + \kappa_x X = 0$$

$$r''_x + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

etc.

Reduction based on identifying

invariants such as

$$\varepsilon_x^2 = 16 \left[ \langle \tilde{x}^2 \rangle_{\perp} \langle \tilde{x}'^2 \rangle_{\perp} - \langle \tilde{x} \tilde{x}' \rangle_{\perp}^2 \right]$$

helps understand solutions

♦ Compact expressions

♦ Easy to formulate

- Straightforward to incorporate additional effects

♦ Natural fit to numerical routine

♦ Easy to code

SM Lund, USPAS, June 2008    Transverse Centroid and Envelope Descriptions of Beam Evolution    85

Please do not remove author credits in any redistributions of class material.

SM Lund, USPAS, June 2008    Transverse Centroid and Envelope Descriptions of Beam Evolution    86

### **References:** For more information see:

#### Image charge couplings:

E.P. Lee, E. Close, and L. Smith, Nuc. Inst. And Methods, 1126 (1987)

#### Seminal work on envelope modes:

J. Struckmeier and M. Reiser, *Theoretical Studies of Envelope Oscillations and Instabilities of Mismatched Intense Charged-Particle Beams in Periodic Focusing Channels*, Part. Accel. **14**, 227 (1984)

M. Reiser, *Theory and Design of Charged Particle Beams* (John Wiley, 1994, 2008)

#### Extensive review on envelope instabilities:

S.M. Lund and B. Bulkh, *Stability Properties of the KV Envelope Equations Describing Intense Ion Beam Transport*, PRSTAB **7** 024801 (2004)

#### Efficient, Fail-Safe Generation of Matched Envelope Solutions:

S.M. Lund and S.H. Chilton, and E.P. Lee, *Efficient Computation of Matched Solutions of the KV Envelope Equations*, PRSTAB **9** 064201 (2006)

SM Lund, USPAS, June 2008    Transverse Centroid and Envelope Descriptions of Beam Evolution    87

These slides will be corrected and expanded for reference and any future editions of the US Particle Accelerator School class:  
**Beam Physics with Intense Space Charge**, by J.J. Barnard and S.M. Lund

### Reduced Equations

Corrections and suggestions are welcome. Contact:

Steven M. Lund  
 Lawrence Berkeley National Laboratory  
 BLDG 47 R 0112  
 1 Cyclotron Road  
 Berkeley, CA 94720-8201

etc.

♦ Easy to formulate

- Straightforward to incorporate additional effects

♦ Natural fit to numerical routine

♦ Easy to code

Please do not remove author credits in any redistributions of class material.

SM Lund, USPAS, June 2008    Transverse Centroid and Envelope Descriptions of Beam Evolution    86

### KV results:

F. Sacherer, *Transverse Space-Charge Effects in Circular Accelerators*, Univ. of California Berkeley, Ph.D Thesis (1968)

I. Kaphinskij and V. Vladimirkij, in *Proc. Of the Int. Conf. On High Energy Accelerators, (American Institute of Physics, 1982)*, AIP Conf. Proc. No. 87, p. 147

*and Instrumentation* (CERN Scientific Info. Service, Geneva, 1959) p. 274

#### Symmetries and phase-amplitude methods:

A. Drago, *Lectures on Nonlinear Orbit Dynamics in Physics of High Energy Particle Accelerators*, (American Institute of Physics, 1982), AIP Conf. Proc. No. 87, p. 147

E. D. Courant and H. S. Snyder, *Theory of the Alternating-Gradient Synchrotron*, Annals of Physics **3**, 1 (1958)

#### Analytical analysis of matched envelope solutions and transport scaling:

E. P. Lee, *Precision matched solution of the coupled beam envelope equations for a periodic quadrupole lattice with space-charge*, Phys. Plasmas **9**, 4301 (2005)

O.A. Anderson, *Accurate Iterative Analytic Solution of the KV Envelope Equations for a Matched Beam*, PRSTAB, **10** 034202 (2006)

SM Lund, USPAS, June 2008    Transverse Centroid and Envelope Descriptions of Beam Evolution    88

### Coupled Moment Formulations of Centroid and Envelope Evolution:

J.J. Barnard, H.D. Shay, S.S. Yu, A. Friedman, and D.P. Grote, *Emittance Growth in Heavy-Ion Recirculators*, 1992 PAC Proceedings, Ontario, Canada, p. 229

J.J. Barnard, J. Miller, I. Haber, *Emittance Growth in Displaced Space Charge Dominated Beams with Energy Spread*, 1993 PAC Proceedings, Washington, p. 3612 (1993)

J.J. Barnard, *Emittance Growth from Rotated Quadrupoles in Heavy Ion Accelerators*, 1995 PAC Proceedings, Dallas, p. 3241 (1995)

R.A. Kishik, J.J. Barnard, and D.P. Grote, *Effects of Quadrupole Rotations on the Transport of Space-Charge-Dominated Beams: Theory and Simulations Comparing Linacs with Circular Machines*, 1999 PAC Proceedings, New York, TUP119, p. 1761 (1999)

J.J. Barnard, R.O. Bangert, E. Henestroza, I.D. Kaganovich, E.P. Lee, B.G. Logan, W.R. Meier, D. Rose, P. Sandhanam, W.M. Sharp, D.R. Welch, and S.S. Yu, *A Final Focus Model for Heavy Ion Fusion System Codes*, NIMA **544** 243-254 (2005).

J.J. Barnard and B. Losic, *Envelope Modes of Beams with Angular Momentum*, Proc. 20th LINAC Conf., Monterey, MOE12 (2000)

§9

## Transport Limit Scaling Based on the Matched Beam Envelope Equations for Periodic Focusing Channels

The scaling of the maximum beam current, or equivalently, the maximum permeance  $Q$  that can be transported at a given energy, with a specified focusing technology and lattice is of critical importance in designing optimal transport and acceleration channels. Needed equations can be derived from approximate analytical solutions to the matched beam envelope equations for a given lattice.

Alternatively, numerical solutions of the envelope equations can be evaluated. But analytical solutions are preferable to understand scaling and enable rapid evaluation of design tradeoffs.

As a practical matter, equations derived must be applied to regimes where technology is feasible.

- Magnet Field Limits
- Electron breakdown
- Vacuum

!

Transport limits are inextricably linked to technology. Moreover, higher order stability constraints etc. must also be respected. Treatments of these topics are beyond the scope of this class. Here we present simplified treatments to highlight issues and methods.

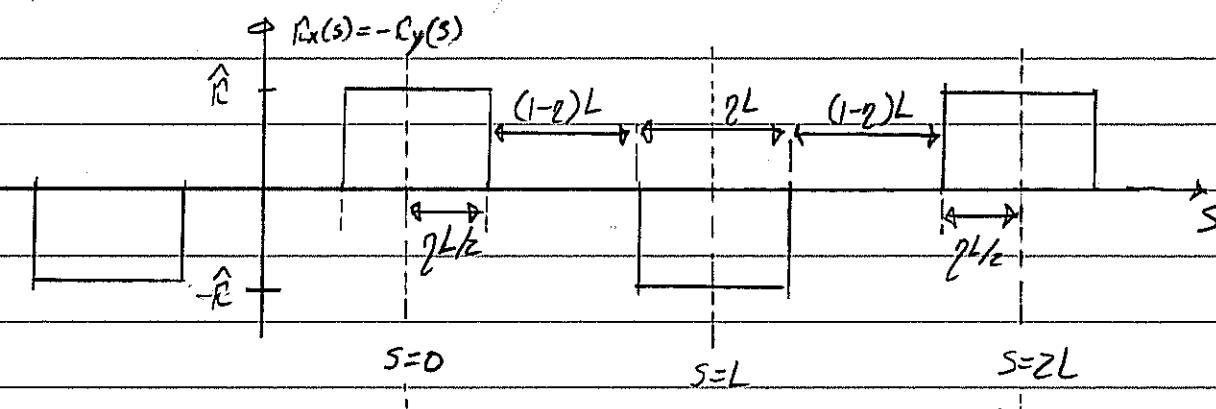
First review an example sketched by J.J. Barnard  
in the Intro. lectures.

### Transport Limits of a Periodic FODO Quadrupole Transport Channel

$$\Gamma_{xm}'' + \frac{(\gamma_b \beta_b)'}{\gamma_b \beta_b} \Gamma_{xm}' + \Gamma_{xm} - 2Q - \frac{E_x^2}{\Gamma_{xm}^3} = 0$$

$$\Gamma_{ym}'' + \frac{(\gamma_b \beta_b)'}{\gamma_b \beta_b} \Gamma_{ym} - E_x \Gamma_{ym} - 2Q - \frac{E_y^2}{\Gamma_{ym}^3} = 0$$

$$\Gamma_{xm}(s + L_p) = \Gamma_{xm}(s) ; \quad \Gamma_{ym}(s + L_p) = \Gamma_{ym}(s)$$



$$L = \text{Half-Period} \quad L = L_p/2$$

$$\eta = \text{Quadrupole "occupancy"} \quad 0 < \eta \leq 1$$

$\eta$  = Focus strength

$$\eta = \begin{cases} \frac{g E_g(s)}{m \gamma_b \beta_b c^2} ; \text{ Electric} \\ \frac{g B_g'(s)}{m \gamma_b \beta_b c} ; \text{ Magnetic} \end{cases}$$

Expand  $R_x(s)$  as a Fourier Series!

$$R_x(s) = \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi s}{L}\right)$$

$$R_n = \frac{1}{L} \int_0^{2L} R_x(s) \cos\left(\frac{n\pi s}{L}\right) ds = \frac{2\hat{R}}{n\pi} \left[ 1 - (-1)^n \right] \sin\left(\frac{n\pi}{2}\right)$$

And expand the periodic matched beam envelope by:

$$f_{xm} = f_b \left[ 1 + \Delta \cos\left(\frac{\pi s}{L}\right) \right] + \sum_{n=2}^{\infty} \Delta x_n \cos\left(\frac{n\pi s}{L}\right)$$

$$f_{ym} = f_b \left[ 1 - \Delta \cos\left(\frac{\pi s}{L}\right) \right] + \sum_{n=2}^{\infty} \Delta y_n \cos\left(\frac{n\pi s}{L}\right)$$

$f_b = \text{const} = \text{avg. beam radius.}$

$|\Delta| = \text{const} \ll 1$

$\Delta x_n$  constants with  $|\Delta x_n| \ll |\Delta|$

Take:

- $(y_b \beta_b)' = 0 \Rightarrow \text{coasting beam}$

- $\varepsilon_x = \varepsilon_y = \varepsilon \Rightarrow \text{isotropic beam}$

and insert these expansions in the envelope equations.

Neglect:

- All terms  $\mathcal{O}(\Delta^2)$  and higher

- Fast oscillation terms  $\sim \cos\left(\frac{n\pi s}{L}\right)$  with  $n \geq 2$ .

to obtain two independent constraint equations:

Avg  $\therefore \frac{2\Delta \hat{R}}{\pi} f_b \sin\left(\frac{\pi \eta}{2L}\right) - \frac{Q}{f_b} - \frac{\varepsilon^2}{f_b^3} = 0$   
 (const)  $\therefore$

Fundamental

$\Delta \cos\left(\frac{\pi s}{L}\right) \therefore -\Delta \left(\frac{\pi}{L}\right)^2 f_b + \frac{4\hat{R} f_b \sin\left(\frac{\pi \eta}{2L}\right)}{\pi} + \frac{3\Delta \varepsilon^2}{f_b^3} = 0$

These equations can be solved to express the maximum beam edge excursion as

$$\text{Max}[\Gamma_{xm}] = \text{Max}[\Gamma_{ym}] \approx r_b(1+|\Delta|) = r_b \left( 1 + \frac{4|\hat{R}|L^2 \sin(\frac{\pi\ell}{2L})}{\pi^3 (1 - \frac{3L^2\varepsilon^2}{\pi^2 r_b^4})} \right)$$

and the beam Pervageance as:

$$Q = \frac{2}{\pi^2} \left[ \frac{\sin(\frac{\pi\ell}{2L})}{(\frac{\pi\ell}{2L})} \right]^2 \frac{\gamma^2 \hat{R} L r_b^2}{\left( 1 - \frac{3L^2\varepsilon^2}{\pi^2 r_b^4} \right)} - \frac{\varepsilon^2}{r_b^2}$$

### Design Strategy:

- 1) Choose a lattice period  $2L$ , quadrupole occupancy  $\gamma$ , and clear machine "pipe" radius  $r_p$  consistent with focusing technology employed.
- 2) Choose the largest possible focus strength  $\hat{R}$  (quadrupole current or voltage excitation) for beam energy with undepressed particle phase advance:

$$\Delta \approx 80^\circ / \text{period.}, \text{"Tiefenbach Limit"}$$

- Larger phase advances correspond to stronger focus and smaller beam cross-sectional area, for given values of  $Q, \varepsilon$ .

- Weaker phase advance suppresses various particle envelope and collective instabilities for reliable transport: [Ref: M.G. Tiefenbach, "Space-Charge Limits on the Transport of Ion Beams," UC Berkeley Ph.D Thesis, 1986 LBL-22465]

- 3) Choose a suitable beam-edge to aperture clearance factor:

$$r_p = \text{Max}[r_m] + \Delta_p$$

$\Delta_p$  = Clearance.

to allow for misalignments, limit scraping of halo particles outside the beam core, reduce image charges, gas propagation times from the aperture to the beam, and other nonideal effects.

- 4) Evaluate choices made using higher-order theory, numerical simulations etc. Iterate choices made to reoptimize when evaluating cost.

Effective application of this formulation requires extensive practical knowledge!

- Nonideal effects: collective instabilities, halo, electron and gas interactions (species contamination), ...
- Technology limits: voltage breakdown, vacuum, superconducting magnets, ....

In practice, for intense beam transport, the emittance terms  $\epsilon_x, \epsilon_y$  can often be neglected for the purpose of obtaining simpler scaling relations that are more easily understood.

$$\lim_{\epsilon_x \rightarrow 0} \delta_x = 0$$

$\Rightarrow$  Full space charge depression

$$\lim_{\epsilon_y \rightarrow 0} \delta_y = 0$$

In this limit  $Q \rightarrow Q_{max}$ , the maximum transportable perecance.

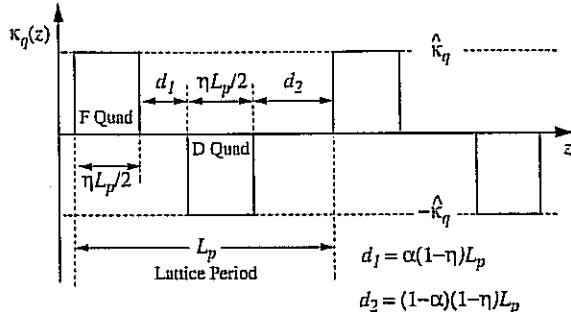
For our previous example for FODO quadrupoles, the  $\epsilon \rightarrow 0$  limit obtains:

$$\lim_{\epsilon \rightarrow 0} \text{Max}[f_m] = R_b \left\{ 1 + \frac{4|\hat{\epsilon}| L^2}{\pi^3} \sin\left(\frac{\pi\eta}{z}\right) \right\}$$

$$\lim_{\epsilon \rightarrow 0} Q = Q_{max} = \frac{z}{\pi^2} \left[ \frac{\sin\left(\frac{\pi\eta}{z}\right)^2}{\left(\frac{\pi\eta}{z}\right)} \right] \eta^2 R_b^2 L^2$$

Unfortunately, the method introduced before are inadequate for lattices with lesser degrees of symmetry such as syncopated quadrupole doublet lattices. However, methods introduced by Lee [E.P. Lee, Physics of Plasmas 9, 4301 (2002)], can be applied in this situation and also obtain more accurate results. It is beyond the scope of this class to carry out derivations with these methods, but we summarize results derived.

### Quadrupole Doublet Lattice

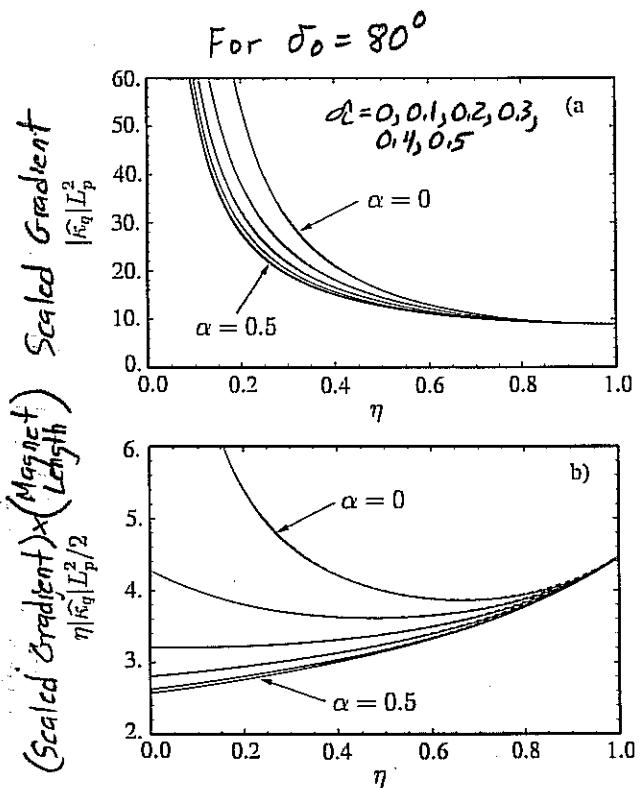


Denote:

$$\text{Avg Radius: } \bar{r}_m = \int_0^{L_p} \frac{ds}{L_p} r_m(s) = \int_0^{L_p} \frac{ds}{L_p} r_{ym}(s)$$

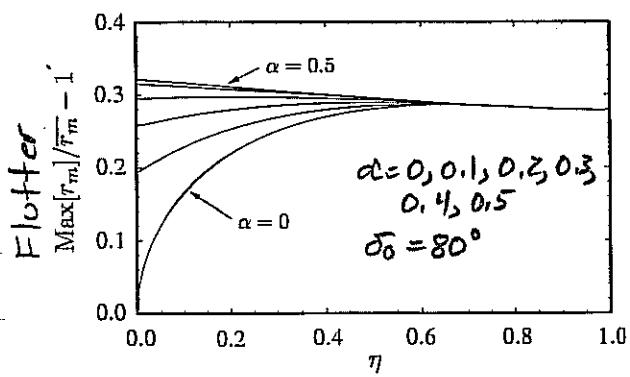
$$\text{Max Excursion: } \underset{\text{in period}}{\text{Max}} [\bar{r}_m] = \underset{\text{in period}}{\text{Max}} [\bar{r}_{xm}, \bar{r}_{ym}]$$

$$\cos \sigma_0 = 1 - \frac{(\eta \kappa_q L_p^2)^2}{32} \left[ \left(1 - \frac{2}{3}\eta\right) - 4 \left(\alpha - \frac{1}{2}\right)^2 (1-\eta)^2 \right].$$



Envelope Flutter

$$\frac{\text{Max}[r_m]}{r_m} - 1 = \frac{(1 - \cos \sigma_0)^{1/2} (1 - \eta/2) [1 - 4(\alpha - 1/2)^2 (1 - \eta)^2]}{2^{3/2} [(1 - 2\eta/3) - 4(\alpha - 1/2)^2 (1 - \eta)^2]^{1/2}}.$$



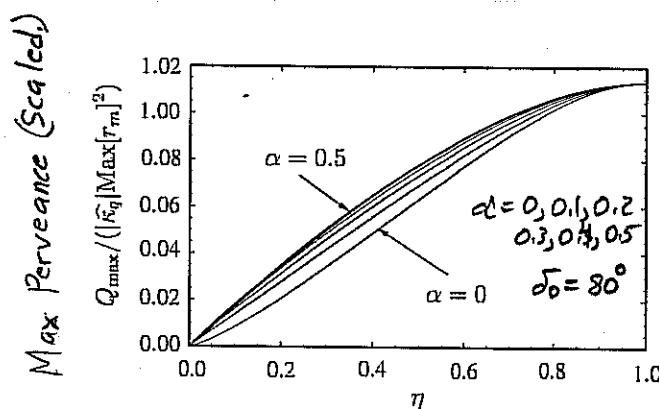
## Relations Connecting Max Transportable Pervance $Q_{\max}$ and Lattice Parameters.

$$Q_{\max} = \frac{(1 - \cos \sigma_0)^{1/2} \eta [(1 - 2\eta/3) - 4(\alpha - 1/2)^2(1 - \eta)^2]^{1/2}}{2^{3/2} (\text{Max}[r_m]/\bar{r}_m)^2} |\kappa_q| \text{Max}[r_m]^2$$

$$= \frac{(1 - \cos \sigma_0)^{1/2} \eta [(1 - 2\eta/3) - 4(\alpha - 1/2)^2(1 - \eta)^2]^{1/2}}{2^{3/2} \left\{ 1 + \frac{(1 - \cos \sigma_0)^{1/2}(1 - \eta/2)[1 - 4(\alpha - 1/2)^2(1 - \eta)^2]}{2^{3/2} [(1 - 2\eta/3) - 4(\alpha - 1/2)^2(1 - \eta)^2]^{1/2}} \right\}^2} |\kappa_q| \text{Max}[r_m]^2.$$

$$\frac{\text{Max}[r_m]}{L_p} = \sqrt{\frac{Q_{\max}}{2(1 - \cos \sigma_0)}} \left( \frac{\text{Max}[r_m]}{\bar{r}_m} \right)$$

$$= \sqrt{\frac{Q_{\max}}{2(1 - \cos \sigma_0)}} \left\{ 1 + \frac{(1 - \cos \sigma_0)^{1/2}(1 - \eta/2)[1 - 4(\alpha - 1/2)^2(1 - \eta)^2]}{2^{3/2} [(1 - 2\eta/3) - 4(\alpha - 1/2)^2(1 - \eta)^2]^{1/2}} \right\},$$



Derivation and application of scaling relations can be complicated. They are often applied in systems codes to generate plots that can be interpreted more readily.